

ON PROBABILITY.

1. In considering any future event, we are generally unable to determine whether or not it will happen; yet, we can often conjecture the number of cases which are possible, and of these how many favour the production of the event in question. In our uncertainty, we say that there is a chance it will happen; and thus our idea of chance arises from our wanting data which might enable us to decide whether or not the event will take place. If, for instance, a bag contain one white and two black balls, it is impossible to decide whether or not a black ball will be drawn in one trial; but we know that there are three cases possible, of which two favour the appearance of a black ball and one the contrary, and of these, we have no reason to think one more probable than another.

2. The operations of the mind are of two kinds; the one consists in acquiring data, the other consists in making deductions from data previously acquired. Our data are only probable; our deductions from these are also probable. The subject, therefore, of this treatise is intimately connected with every science, and, whether on account of its numerous and useful applications, or of the exact reasoning by which its principles are established, carries with it the highest degree of interest. In the sequel we shall explain the method of applying it to the calculation of life annuities, a few tables of which will be subjoined.

3. To avoid circumlocution, those cases which embrace the production of a particular event are called the favourable cases, and those which do not, the unfavourable cases. It is usual to apply the word belief to the past, and the word expectation to the future; but the theory of probability is in all respects the same, whether it be applied to past or to future events. When

we endeavour to discover whether an event $\left\{ \begin{smallmatrix} \text{did} \\ \text{will} \end{smallmatrix} \right\}$ happen, we review the different cases which are possible. If the favourable cases are more numerous than the unfavourable, we $\left\{ \begin{smallmatrix} \text{believe} \\ \text{expect} \end{smallmatrix} \right\}$ that the event $\left\{ \begin{smallmatrix} \text{did} \\ \text{will} \end{smallmatrix} \right\}$ take place.

The words believe and expect, and those to which they correspond, are placed between brackets, in order to show that the reasoning is the same in both cases. Let us suppose that a bag contains one black and two white balls: if I am asked whether a white ball will be drawn, or if, a ball being already drawn, but concealed from my view, I am asked whether a white ball has been drawn, it is clear that the judgment formed in both cases will be the same. I answer in both cases, that it is more probable that the ball which is drawn is white than black, yet if the ball be already drawn, but concealed from my view, the event is already determined and certain. We perceive, therefore, that the estimation of probability has no necessary reference to actual occurrence, but only to the means of judging which a given individual possesses.

4. We have used the word probability before giving its definition, because its popular meaning has hitherto sufficed for our purpose. We now give its mathematical definition, which is this: the probability of any event is the ratio of the favourable cases to all the possible cases which, in our judgment, are similarly circumstanced with regard to their happening or failing. Thus, if a bag contain one white and two black balls, the probability of drawing a white ball is $\frac{1}{3}$; the probability of throwing ace with a die at the first

throw is $\frac{1}{2}$; or, in common language, we should say of the first of these events that the odds are 2 : 1 against it, and of the second 5 : 1.

Generally, if $m + n$ is the whole number of cases, and if m is the number of cases which are favourable to the event P, $m : n$ are the odds
 n unfavourable

in favour of the event P, and the probability of the event P is $\frac{m}{m+n}$

5. Simpson has defined the probability of an event to be the ratio of the chances by which the event in question may happen to all the chances by which it may happen or fail. In this definition the word chance must be understood a way of happening; we, however, frequently say, "I left such a thing to chance," or, "such a thing is entirely chance;" these expressions, which are in some measure sanctioned by common use, are intended to signify that we are ignorant of the causes which produce the event in question, or that we do not influence its occurrences.

6. When there are many events, such that one must, and only one can happen on any given trial, we shall call them conflicting events; and it is evident, from the definition of probability, that the probability that either of two conflicting events will happen on any single trial is equal to the sum of their respective probabilities; it is also evident that the sum of the probabilities of all the conflicting events which can happen on any single trial is expressed by unity; for, by the supposition, one of them must happen.

Our { belief
 expectation } is founded upon the probability of the event under consideration. It will often happen that our judgment is influenced by circumstances of too complicated or delicate a nature to be submitted to numerical calculation; and the conclusions with which this science furnishes us are true only within the limits of the errors which arise from neglecting these considerations. We experience the same difficulty in applying to physical phenomena the theories deduced from abstract principles of measure and motion. When all the cases which are possible, are favourable, the event is certain and belief becomes certitude, or knowledge. *Certainty*, which is the greatest probability, is therefore represented by unity; it must be distinguished from the highest degree of belief which we have called *certitude*; they are often confounded with each other, while they differ in the same manner as probability and belief differ. In fact, we want two words for every stage as much as for unity, the one to express the ratio of the favourable to all the cases possible, the other denoting the opinion consequent on the perception of that ratio.

7. If a bag contain no white and ten black balls, the probability of drawing a white ball is $\frac{0}{10}$, or, zero; if, on the other hand, the bag contain 10 white and no black balls, the probability of drawing a white ball is $\frac{10}{10}$, or unity, and whatever be the number of black balls, the probability of drawing a white ball must be some fraction between 0 and 1, which are its limits. When the fraction which expresses the probability of an event is little different from unity, we say the event is very probable, or nearly certain; when it is but little greater than $\frac{1}{2}$, we say it is probable; when $\frac{1}{2}$, doubtful; when rather less than $\frac{1}{2}$, improbable; when much less than $\frac{1}{2}$, very improbable; and when zero, impossible.

8. We habitually assent to propositions which have in their favour a probability less than unity: this degree of probability is vulgarly called moral certainty, an expression which is at variance with every analogy of language. The state of mind of a man who is aware of unfavourable events

which are possible, but who disregards them by reason of their reputed improbability, is perhaps what is meant. Some philosophers have endeavoured to fix the numerical fraction to which this moral certainty is equal by observing the risks of which men are in general careless. Buffon chose the fraction $\frac{20000}{100000}$; Condorcet estimated it in a different manner, and of course obtained a very different result. Indeed it is obvious that this fraction is arbitrary, and we shall therefore not enter more minutely into this question. There may, perhaps, be a practical utility for each man to determine the risk his own temperament enables him to disregard, in order to obtain a standard with which to compare the results of occasional theorems: without some such comparison they might fail in their abstract numerical form to determine his judgment.

9. We have said that probability does not exist in the abstract, but always refers to the knowledge possessed by some particular individual. Let us suppose that a bag contains one white and two black balls, and that A having drawn a white ball holds it so that he can see what colour it is, but so that B cannot. Here three cases appear possible to B, of which two favour the drawing a black ball; the probability therefore that it is a black ball to B is $\frac{2}{3}$, while the probability to A that it is a white ball, is unity, or certainty. Again: suppose a bag contain one white, one black, and one red ball; A, having drawn a white ball, whispers to B that the ball which is drawn is not red. Three cases appear equally possible to C, of which one only favours the drawing a white ball; C therefore estimates the probability at $\frac{1}{3}$, while B (if he believes the information given him by A) has only two alternatives to choose between: he therefore estimates the probability at $\frac{1}{2}$. Even if B do not implicitly believe the information given him by A, it is clear that his judgment will be formed on grounds different from those on which C decides.

10. It is thus that the same fact related before a numerous audience obtains from different individuals different degrees of belief: this is chiefly to be attributed to the different degrees of knowledge possessed by different individuals of circumstances which bear on the fact in question. An inhabitant of the torrid zone has difficulty in believing that water freezes; and the recovery of a sick person may appear probable to one unacquainted with medicine, while the skilful physician despairs of effecting a cure.

11. It follows from the definition of probability, that to determine the probability of any event, it is only necessary to enumerate the cases which are favourable and those which are unfavourable to its production, in order to form the fraction which expresses its probability. In order that this may be well understood, we shall begin with some very simple examples, and for these it will be necessary, at first, to have recourse to games of chance, in which the whole number of possible occurrences is most readily ascertained.

Ex. 1. Suppose a piece of money is thrown into the air, and that the probability of its falling on the obverse side twice successively is required: here the following cases present themselves.

Case 1. The obverse both times.

2. The obverse the first time and the reverse the second.

3. The reverse the first time and the obverse the second.

4. The reverse both times.

These are the only cases possible; and if we are ignorant of the existence of any cause tending to make the piece fall on one side rather than on the other, they are all similarly circumstanced, and therefore the probability of each case is $\frac{1}{4}$.

Since every one of the six numbers on one of the dice may combine with every one of the six on the other, the number of throws on the dice is 36.

The number 7 may be made up of $\left\{ \begin{array}{l} 1 \text{ and } 6 \\ \text{or } 3 \text{ and } 4 \\ \text{or } 2 \text{ and } 5 \end{array} \right\}$,

and as these numbers may be on the one die or the other, there are in all six ways which favour the number 7, and therefore the probability required is $\frac{6}{36}$ or $\frac{1}{6}$.

Ex. 4. A, the dealer in a party at whist, desires to know the probability of his partner holding a given card. The number of cards which are held by the other three players is 39; therefore the probability that the card in question is any given card in A's partner's hand is $\frac{1}{39}$, but it may be any one of the 13 cards which A's partner holds, therefore the probability is $\frac{13}{39} = \frac{1}{3}$. Or thus, there are three cases possible: either the card is in the hand of A's partner, or of one of the other two players; and as these three cases are similarly circumstanced, the probability of either of them is $\frac{1}{3}$, the odds against it being of course 2 : 1.

A desires to know the probability of his partner holding 2 given cards.

The number of combinations of 39 things taken two and two together is $\frac{39 \times 38}{1.2}$, therefore the probability that these two cards are any given two

cards in A's partner's hand is $\frac{1}{\frac{39 \times 38}{1.2}} = \frac{1}{39 \times 19}$; but they may be any

two cards in A's partner's hand; therefore, since the number of combinations of 13 cards taken two and two together, is $\frac{13 \times 12}{1.2} = 13 \times 6$,

the probability required is $\frac{13 \times 6}{39 \times 19} = \frac{2}{19}$, the odds against are therefore 17 : 2.

Similarly, the probability that he holds any three given cards, is $\frac{22}{703}$; the odds against are, therefore, 681 : 22.

Ex. 5. Required the probability, that in a deal at whist each player holds an honour.

The number of permutations of 52 cards taken all together is $52 \times 51 \dots \times 3 \times 2 \times 1$, and the number of permutations of 13 cards taken all together is $13 \times 12 \times 11 \dots \times 3 \times 2 \times 1$, therefore the number of different deals is

$$\frac{52 \times 51 \times 50 \dots \times 3 \times 2 \times 1}{(13 \times 12 \times 11 \dots \times 3 \times 2 \times 1)^4},$$

because the 13 cards may be permuted in each player's hand separately, without altering his hand.

The number of permutations of 48 cards taken all together is $48 \times 47 \times 46 \dots \times 3 \times 2 \times 1$, therefore the number of different ways in which 48 cards can be dealt to four persons is

$$\frac{48 \times 47 \times 46 \dots \times 3 \times 2 \times 1}{(12 \times 11 \times 10 \dots \times 3 \times 2 \times 1)^4}.$$

Let A, B, C, D be the hands dealt to each player out of 49 cards,

a, b, c, d , the four honours.

It is evident that A, B, C, D may be combined with a, b, c, d in as many different ways as a, b, c, d can be permuted, that is, in $4 \times 3 \times 2 \times 1$, or 24 different ways; therefore the probability that

$$\begin{aligned} \text{each hand has an honour} &= \frac{48 \times 47 \times 46 \dots 3 \times 2 \times 1}{(12 \times 11 \times 10 \dots 3 \times 2 \times 1)^4} \times 24. \\ &= \frac{13^4}{52 \times 51 \times 50 \times 49} \times 24 = \frac{2197}{20825}, \quad \begin{array}{l} \text{Odds against.} \\ \text{About} \end{array} 17 : 2 \end{aligned}$$

So it may be found that the probability that

$$\begin{aligned} \left\{ \begin{array}{l} \text{one hand has the four} \\ \text{honours} \end{array} \right\} &= \frac{13 \times 12 \times 11 \times 10}{52 \times 51 \times 50 \times 49} \times 4 = \frac{220}{20825}, & 94 : 1 \\ \left\{ \begin{array}{l} \text{two hands have each} \\ \text{two honours} \end{array} \right\} &= \frac{13 \times 12 \times 13 \times 12}{52 \times 51 \times 50 \times 49} \times 6 \times 6 = \frac{2808}{20825}, & 13 : 2 \\ \left\{ \begin{array}{l} \text{one hand has two ho-} \\ \text{nours and two hands one} \end{array} \right\} &= \frac{13 \times 13 \times 13 \times 12}{52 \times 51 \times 50 \times 49} \times 12 \times 12 = \frac{12168}{20825}, & 5 : 7 \\ \left\{ \begin{array}{l} \text{one hand has three ho-} \\ \text{nours and one hand one} \end{array} \right\} &= \frac{13 \times 13 \times 12 \times 11}{52 \times 51 \times 50 \times 49} \times 12 \times 4 = \frac{3432}{20825}, & 5 : 1 \end{aligned}$$

The number of deals essentially different is

$$\frac{52 \times 51 \times 50 \dots \times 3 \times 2 \times 1}{(13 \times 12 \times 11 \dots 3 \times 2 \times 1)^4} \times \frac{1}{24}.$$

This number, of which the logarithm is 27.34935, is so great, that if 1,000,000,000 persons, about the population of the earth, were to deal the cards incessantly day and night for 100,000,000 years, at the rate of a deal by each person a minute, and even if each of these deals were essentially different, they would not have exhausted a $\frac{1}{7000000}$ th part of the number of essentially different ways in which 52 cards can be distributed equally between four players.

12. Let a bag contain 10 slips of paper, each having the name of a different individual written upon it, and suppose that the individual whose name is drawn is to receive 10*l*.

In estimating the value of the expectation of these individuals it is evident that the sum of the values of all their expectations is equal to 10*l*., the sum which one of them must receive, for if they each sold their chance of winning the 10*l*. to another person, this person would be sure of receiving 10*l*.; and since their expectations are equal, for their chances of winning are the same, if e be the expectation of one of them

$$10e = 10, \quad e = 1$$

that is, the expectation of each is worth 1*l*. It is evident, also, that if there were $m + n$ slips of paper, each having the name of a different individual written upon it, and if the individual whose name was drawn should receive a pounds, the value of the expectation of each individual would be

$$\frac{a}{m+n}. \quad \text{The value of the expectation of two individuals is the sum of the}$$

values of their separate expectations, and is, therefore, $\frac{2a}{m+n}$; of three individuals $\frac{3a}{m+n}$; and of m individuals is $\frac{ma}{m+n}$; but the expectation of m individuals must be the same as the expectation of one individual holding m tickets; therefore, the expectation of an individual holding m tickets is $\frac{ma}{m+n}$, or $\frac{m}{m+n} \times a$, but $\frac{m}{m+n}$ is the probability of this individual winning; therefore, generally, the value of the expectation of any individual with respect to any particular event is the product of the probability that the event will take place by the gain that will accrue to the individual if it does take place.

Ex. 6. Suppose the probability that a horse wins a race is p : A gives B £ a : what odds should B bet A that the horse wins, in order that, on the whole, their situation may be the same? Let $x:1$ be the odds required; then the expectation of B $= a + p - (1-p)x$: the expectation of A $= (1-p)x - a - p$; in order that they may be in the same situation, $a + p - (1-p)x = (1-p)x - (a + p)$; $x = \frac{a+p}{1-p}$.

13. All wagers are founded upon this principle, of multiplying the probability of the event by the contingent gain. Generally, however, one party does not pay down to the other a sum of money in order to receive another sum of money if a certain event takes place; but A engages to pay B a certain sum if the event takes place, and B engages to pay A a certain sum if it does not take place. Let a be the sum which A is to pay B, on the event of which the probability is p ; and b the sum to be paid by B to A, if that happens of which the probability is q .

The expectation of A is equal to $bq - ap$.

The expectation of B is equal to $ap - bq$.

In order that the wager may be fair, the expectations of A and B must be equal.

$$bq - ap = ap - bq \quad ap = bq$$

$$\frac{a}{b} = \frac{q}{p} \cdot \frac{b}{a+b} = p, \quad \frac{a}{a+b} = q, \text{ since } p+q=1: a \text{ and } b \text{ are the odds.}$$

Since the sum of the probabilities of any number of conflicting events is equal to unity, we have an equation of condition between the odds; and whenever they do not satisfy this equation, it is possible to bet with the certainty of gain.

Ex. 7. Suppose three horses, A, B, C, are entered for a race, and that I bet £12. to £5. against A; £11. to £6. against B; and £10. to £7. against C.

If A wins, I gain £6. + 7 - 12 = £1.

B wins, I gain £5. + 7 - 11 = £1.

C wins, I gain £5. + 6 - 10 = £1.

Thus I gain £1. whichever horse wins, from having taken the field against each horse.

Here the odds quoted in favour of $\left\{ \begin{array}{l} \text{A are } 5:12 \\ \text{B } \text{ " } 6:11 \\ \text{C } \text{ " } 7:10 \end{array} \right\}$ the corresponding probability would be $\left\{ \begin{array}{l} \frac{5}{17} \text{ in favour of A.} \\ \frac{6}{17} \text{ " " " B.} \\ \frac{7}{17} \text{ " " " C.} \end{array} \right\}$

$$\frac{5}{17} + \frac{6}{17} + \frac{7}{17} = \frac{18}{17} = 1 + \frac{1}{17}$$

The odds are often, as in this case, far from satisfying the equation of condition to which we have alluded.

The odds quoted just before the race upon the horses entered for the Oaks Stakes,* May, 1828, were so enormous, that the corresponding probability

of one of the first seven favourites winning, was $1 + \frac{31}{336}$, exclusive of the winner and six others which started.

14. Although it is easy in similar cases to see whether or not to *back the field*, it is not possible to determine the proportions which should be betted against each event in order that the advantage may be the greatest, unless the probabilities of each are known; and it may be observed, that the odds offered and taken will sometimes depend upon the incertitude of the better, whether or not he will receive the amount of his wager in case of a favourable issue. This chance enters as legitimately into the estimation of the odds as any other, though it may be more difficult to estimate its influence. It may also happen that the odds will be affected by those who, having almost secured a certain advantage in the manner we have just described, will prefer to diminish that advantage by offering more than the real odds against the events which yet remain out of their list, rather than leave any contingency unprovided against. But this last consideration verges on a branch of the subject which we shall afterwards treat more in detail: sufficient reason has been given why, practically, the odds vary so much from the result of a theory, which has not estimated these causes of discrepancy.

15. Let P_1, Q_1 represent two conflicting events.

P_2, Q_2 any other two conflicting events.

and let m_1, m_2, n_1, n_2 be the number of cases favourable to the event $\left\{ \begin{matrix} P_1 \\ P_2 \\ Q_1 \\ Q_2 \end{matrix} \right\}$

P_1 cannot happen with Q_1 , nor P_2 with Q_2 : the whole number of cases possible, therefore, is $(m_1 + n_1)(m_2 + n_2)$. Of these

$\left\{ \begin{matrix} m_1 m_2 \\ m_1 n_2 \\ n_1 m_2 \\ n_1 n_2 \end{matrix} \right\}$ are favourable to the pro- $\left\{ \begin{matrix} (P_1 P_2) \\ (P_1 Q_2) \\ (Q_1 P_2) \\ (Q_1 Q_2) \end{matrix} \right\}$ that is, of the events $\left\{ \begin{matrix} P_1, P_2 \\ P_1, Q_2 \\ Q_1, P_2 \\ Q_1, Q_2 \end{matrix} \right\}$

simultaneously, and therefore by previous definition

$$\left\{ \begin{matrix} \frac{m_1 m_2}{(m_1 + n_1)(m_2 + n_2)} \\ \frac{m_1 n_2}{(m_1 + n_1)(m_2 + n_2)} \\ \frac{n_1 m_2}{(m_1 + n_1)(m_2 + n_2)} \\ \frac{n_1 n_2}{(m_1 + n_1)(m_2 + n_2)} \end{matrix} \right\} \text{ is the probability of the event } \left\{ \begin{matrix} (P_1, P_2) \\ (P_1, Q_2) \\ (Q_1, P_2) \\ (Q_1, Q_2) \end{matrix} \right\}$$

* Betting for the Oaks, Morning Chronicle, May 24, 1828.

5 to 2 against *Ridicule*.

5 . 2 .. *Zoe*.

4 . 1 .. *Rosalie*.

7 . 1 .. *Trampoline*.

14 . 1 .. *Delta*.

14 to 1 against *L'Estelle*.

15 . 1 .. *Ruby*.

25 . 1 .. *Turquoise*, who won.

6 others started.

$\left. \begin{array}{l} \frac{m_1}{m_1 + n_1} \\ \frac{m_2}{m_2 + n_2} \end{array} \right\}$ is the probability of the event $\left\{ \begin{array}{l} P_1 \\ P_2 \end{array} \right\}$ considered by itself,

and as P_1, P_2 may represent any independent events whatever, we conclude that the probability of the concurrence of any two independent events is equal to the product of the probabilities of each considered separately.

It is easy to extend the same theorem to any number of independent events: for we may consider the event (P_1, P_2) as one event of which the

probability is $\frac{m_1 m_2}{(m_1 + n_1)(m_2 + n_2)}$, and if $\frac{m_3}{m_3 + n_3}$ be the probability of any

other independent event P_3 , the probability of the concurrence of (P_1, P_2) and

P_3 or (P_1, P_2, P_3) is equal to $\frac{m_1 m_2}{(m_1 + n_1)(m_2 + n_2)} \times \frac{m_3}{m_3 + n_3}$,

or $\frac{m_1 m_2 m_3}{(m_1 + n_1)(m_2 + n_2)(m_3 + n_3)}$; but $\frac{m_1}{m_1 + n_1}$, $\frac{m_2}{m_2 + n_2}$, and $\frac{m_3}{m_3 + n_3}$, are

the probabilities of the events P_1, P_2, P_3 , considered separately.

The same reasoning may be extended to any number of independent events, and hence this general and important theorem: the probability of the concurrence of any number of independent events is equal to the product of the probabilities of each considered separately. Before we proceed farther, it may be well to illustrate this by an example.

Ex. 8. Let us assume p_1 to be the veracity of any witness A_1 , or the probability that A_1 tells the truth; p_2 the veracity of any other witness A_2 . A_2 asserts that A_1 has asserted that a certain event took place; what is the probability that it did take place?

The event took place, if both A_1 and A_2 tell the truth; the probability of this is $p_1 \times p_2$.

It also took place if both A_1 and A_2 lie, that is, if A_1 said it did not take place, and if A_2 says that A_1 said it did take place: the probability of this is $(1 - p_1)(1 - p_2)$; and the probability that the event did take place, is $p_1 p_2 + (1 - p_1)(1 - p_2)$, or $1 - p_1 - p_2 + 2 p_1 p_2$.

Suppose, for instance, p_1 and p_2 were equal to $\frac{7}{10}$ ths, the probability that the event took place which A_1 , on the authority of A_2 , asserts to have taken place, would be $\frac{49}{100}$ ths, a probability which is considerably less than $\frac{7}{10}$ ths.

The same might be extended to any number of testimonies; and it is thus that events, probable in the first instance, by passing through many relaters, may at last become extremely improbable. Laplace compares this diminution of probability to the diminution of light in passing through a succession of transparent mediums: the analogy does not strike us as being very forcible.

16. Since the odds are inversely as the probabilities of the events, whenever the odds in favour of two independent events are known, the odds that they will both happen may be found.

Ex. 9. In May, 1828, the odds were 3 to 1 against Rapid Rhone winning the Gold Cup, and 6 to 1 against Bessy Bedlam winning the St. Leger. According to these odds, the probability of Rapid Rhone winning the Gold Cup is $\frac{1}{4}$, and of Bessy Bedlam winning the St. Leger is $\frac{1}{7}$. Therefore, the probability of both these events taking place is $\frac{1}{28}$, and the odds against it 27 to 1, or about 1000 to 37. The odds were given at 1000 to 60.

It may be remarked that if the *same* horse as Rapid Rhone were to run in both races, and if the probability of his winning the first race were $\frac{1}{2}$, and of his winning the second $\frac{1}{2}$; the probability of his winning both is not $\frac{1}{2}$, but rather more. This very important distinction will be more fully explained in treating of probabilities *à posteriori*.

17. Let P, Q , be any two conflicting events, of which the probabilities are p, q .

The probability of the concurrence of all the events $P, P, \dots P$, is $p, \times p, \dots p$, and the probability of any other event is that term in the product

$$(p + q_1)(p_2 + q_2) \dots (p_n + q_n);$$

in which the indices of p and q are the same as those of P and Q in the composite event which is considered.

If we consider repeated trials of the same event, so that $P, P, \dots P$, are all the same, $(p_1 + q_1)(p_2 + q_2) \dots (p_n + q_n)$ becomes $(p + q)^n$, and the probability of any event composed of a times P and b times Q in any given order is $p^a q^b$, the probability of having a times P and b times Q , without any regard to the order in which they occur, is the sum of the terms which are equal to $p^a q^b$ in the development of $(p + q)^n$, or, what is the same thing, the term which has $p^a q^b$ for its argument in the expansion of $(p + q)^n$.

By the binomial theorem, this term is $\frac{n \cdot (n-1) \dots \dots \dots 1}{1 \cdot 2 \cdot 3 \dots a \dots 1 \cdot 2 \cdot 3 \dots b} \times p^a q^b$.

Ex. 10. Thus, if n shillings are thrown into the air, in order to find what is the probability of any particular combination, it is only necessary to take

the corresponding term in the development of $\left\{\frac{1}{2} + \frac{1}{2}\right\}^n$.

Suppose there are five shillings

$$\left\{\frac{1}{2} + \frac{1}{2}\right\}^5 = \left(\frac{1}{2}\right)^5 + 5 \cdot \left(\frac{1}{2}\right)^4 + \frac{5 \cdot 4}{1 \cdot 2} \left(\frac{1}{2}\right)^3 + \frac{5 \cdot 4}{1 \cdot 2} \left(\frac{1}{2}\right)^2 + 5 \cdot \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^5.$$

The probability that they will all fall heads = $\left(\frac{1}{2}\right)^5$

$$4 \text{ heads and } 1 \text{ tail} = 5 \times \left(\frac{1}{2}\right)^5$$

$$3 \text{ heads and } 2 \text{ tails} = 10 \times \left(\frac{1}{2}\right)^5$$

$$2 \text{ heads and } 3 \text{ tails} = 10 \times \left(\frac{1}{2}\right)^5$$

$$1 \text{ head and } 4 \text{ tails} = 5 \times \left(\frac{1}{2}\right)^5$$

$$5 \text{ tails} = \left(\frac{1}{2}\right)^5$$

18. The same theorem may be extended to any number of conflicting events, so that if P, Q, R, S represent any conflicting events of which the probabilities are p, q, r, s , respectively, the probability of the event which is composed of a times P, b times Q, c times R, d times S , &c. in $a + b + c + d$ repeated trials is the term in the expansion of $(p + q + r + s)^{a+b+c+d}$, which has $p^a q^b r^c s^d$ for its argument; by the multinomial theorem this term is

$$\frac{(a + b + c + d)(a + b + c + d - 1) \dots \dots 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \dots a \cdot 1 \cdot 2 \cdot 3 \dots b \cdot 1 \cdot 2 \cdot 3 \dots c \cdot 1 \cdot 2 \cdot 3 \dots d} p^a q^b r^c s^d.$$

Ex. 11. Let a jury be composed of n jurymen, and let p be the probability that each jurymen separately will give a right decision, q the probability that he will give a wrong decision. The probability of a unanimous verdict, that is, that they all will voluntarily give a wrong or all a right decision, is $\frac{p^n + q^n}{(p+q)^n}$:

if $p = \frac{9}{10}$, $q = \frac{1}{10}$ and $n = 12$, this is equal to $\frac{9^{12} + 1^{12}}{10^{12}}$, or about $\frac{1}{1000}$.

This probability must not be confounded with the probability, after an unanimous decision has been given, that it is a correct one, which is a very different question.

Ex. 12. Let A and B be two gamblers, and let p be the probability of A's winning a game, and q the probability of B's winning a game; required the probability of A's winning m games out of $m+n$, the set being supposed to finish as soon as A has won m games.

The probability of A's winning the first m games running, is p^m . The probability of A's winning m games out of $m+1$, is $(m+1)p^m q$. The probability of A's winning the first m games, and B the $(m+1)^{\text{th}}$ out of $m+1$ games, is $p^m q$; therefore, the probability of A's winning the set in exactly $m+1$ games is $(m+1)p^m q - p^m q$, or $(m+1-1)p^m q = m p^m q$, and the probability of A's winning the set in not more than $m+1$ games is $p^m + m p^m q$.

The probability of A's winning m games out of $m+2$, in any order, is

$$\frac{(m+2) \cdot (m+1)}{1 \cdot 2} p^m q^2,$$

but the probability of A's winning m games in the first $m+1$ is $(m+1) \cdot p^m q$, and the probability of B's winning the $(m+2)^{\text{th}}$ game is q ; therefore, the probability of A's winning m games in the first $m+1$ out of $m+2$ games is $(m+1) \cdot p^m q^2$; and the probability of A's winning the set exactly in $m+2$ games is

$$\left\{ \frac{(m+2) \cdot (m+1)}{1 \cdot 2} - (m+1) \right\} p^m q^2 = \frac{m \cdot (m+1)}{1 \cdot 2} p^m q^2;$$

and the probability of A's winning the set in not more than $m+2$ games is

$$p^m + m p^m q + \frac{m \cdot (m+1)}{1 \cdot 2} p^m q^2.$$

The same reasoning may be applied to the general term: thus the probability that A will win m games out of $m+n$ in any order

$$= \frac{(m+n) \cdot (m+n-1) \cdot \dots \cdot 2 \cdot 1}{1 \cdot 2 \cdot \dots \cdot m \cdot 1 \cdot 2 \cdot \dots \cdot n} p^m \cdot q^n.$$

The probability that A will win m games out of $m+n-1$, and that B will win the $(m+n)^{\text{th}}$

$$= \frac{(m+n-1) \cdot (m+n-2) \cdot \dots \cdot 2 \cdot 1}{1 \cdot 2 \cdot \dots \cdot m \cdot 1 \cdot 2 \cdot \dots \cdot (n-1)} p^m \cdot q^n.$$

Therefore the probability that A will win m games in exactly $m+n$ games

$$\begin{aligned} &= \frac{(m+n-1) \cdot (m+n-2) \cdot \dots \cdot 2 \cdot 1}{1 \cdot 2 \cdot \dots \cdot m \cdot 1 \cdot 2 \cdot \dots \cdot (n-1)} \left\{ \frac{m+n}{n} - 1 \right\} p^m q^n \\ &= \frac{m \cdot (m+1) \cdot \dots \cdot (m+n-1)}{1 \cdot 2 \cdot \dots \cdot n} p^m q^n, \end{aligned}$$

and the probability of A's winning the set in not more than $m + n$ games will appear to be

$$p^m \left\{ 1 + m q + \frac{m \cdot (m+1)}{1 \cdot 2} q^2 \dots \frac{m \cdot (m+1) \cdot (m+2) \dots (m+n-1)}{1 \cdot 2 \cdot 3 \dots n} q^n \right\}.$$

If A, in order to win the set, must win m games before B wins n games, A must win m games out of $m + n - 1$, and the probability of this event is

$$p^m \left\{ 1 + m q + \frac{m \cdot (m+1)}{1 \cdot 2} q^2 \dots \frac{m \cdot (m+1) \cdot (m+2) \dots (m+n-2)}{1 \cdot 2 \cdot 3 \dots (n-1)} q^{n-1} \right\},$$

and the probability of B's winning n games out of $m + n - 1$ is

$$q^n \left\{ 1 + n p + \frac{n \cdot (n+1)}{1 \cdot 2} p^2 \dots \frac{n \cdot (n+1) \cdot (n+2) \dots (n+m-2)}{1 \cdot 2 \dots 3 \dots (m-1)} p^{m-1} \right\}.$$

The same result may be obtained from the following considerations. If the play be supposed to continue without end, the probability that A will gain a single game

$$= p + p q + p q^2 + \dots \infty = \frac{p}{1-q}, \text{ or } 1,$$

and the probability that A will win any finite number of games as m will be represented by $\left(\frac{p}{1-q} \right)^m$

$$= p^m \left\{ 1 + m q + \frac{m \cdot (m+1)}{1 \cdot 2} q^2 + \dots \infty \right\}.$$

This probability is made up of the partial probabilities that A will win m games in m exactly, in $m + 1$ exactly, &c. The probability that A will win m games in $m + x$ exactly must have $p^m \cdot q^x$ for its argument, and, therefore, since the above-written series contains all these partial probabilities, and no others, and consists solely of terms whose arguments are of the form $p^m \cdot q^x$, each of these partial probabilities will be rightly represented by the corresponding term in that series, for it can be exhibited in no other shape, having the arguments of all its terms of this necessary form. Therefore the probability that A will win m out of $m + n$ games consists of the first terms of that series up to that inclusively, whose argument is $p^m \cdot q^n$, the same as before obtained.

If A wants m games of being up, and B n games, and they agree to leave off playing, the stakes should be divided between them in the proportions of their probabilities.

This problem is celebrated in the history of the theory of probabilities, and was the first of any difficulty which was solved. It was proposed to Pascal by the Chev. de Méré, with some others relating to games with dice.

Ex. 13. A bag contains $n + 1$ balls which are marked by the numbers $0, 1, 2, 3, \dots, n$, a ball is drawn and afterwards replaced in the bag. Required the probability that after i drawings the sum of the numbers drawn is equal to s .

A little consideration will show that the probability required will be the coefficient of x^s in the expansion of $\left\{ \frac{x^0 + x^1 + x^2 \dots + x^n}{n+1} \right\}^i$ because that coefficient will be made up of all the different ways in which the different indexes of x can be combined in this development, so as to equal the required index s .

$$x^s + x^{s-1} + x^{s-2} + \dots + x^0 = \frac{1 - x^{s+1}}{1 - x}$$

$$\{1 - x^{s+1}\}^i = 1 - i x^{s+1} + \frac{i(i-1)}{1.2} x^{2(s+1)} - \frac{i(i-1)(i-2)}{1.2.3} x^{3(s+1)} + \&c.$$

$$\{1 - x\}^{-i} = 1 + i x + \frac{i(i+1)}{1.2} x^2 + \frac{i(i+1)(i+2)}{1.2.3} x^3 + \&c.$$

The coefficient of x^s is obtained by multiplying the coefficient of the first term of the upper series by the coefficient of x^s in the lower, that of the second in the upper by that of x^{s-1} in the lower, &c. and

$$= \frac{i(i+1)(i+2)\dots(i+s-1)}{1.2.3\dots s(n+1)^i} - i \frac{i(i+1)(i+2)\dots(i+s-n-2)}{1.2.3\dots(s-n-1)(n+1)^i} \\ + \frac{i(i-1)i(i+1)(i+2)\dots(i+s-2n-3)}{1.2\dots(s-2n-2)}(n+1)^i \&c$$

This series is equivalent to

$$\frac{(s+1)(s+2)\dots(s+i-1)}{1.2\dots(i-1)(n+1)^i} - i \frac{(s-n)(s-n+1)\dots(s-n+i-2)}{1.2\dots(i-1)(n+1)^i} \\ + \frac{i(i-1)}{1.2} \frac{(s-2n-1)(s-2n)\dots(s-2n+i-3)}{1.2\dots(i-1)(n+1)^i} \&c.$$

It is to be remarked, that all the terms of the development of $(1 - x^{s+1})^i$ in which x is involved to a higher power than s may be rejected, because we have no negative powers of x in the other factor $(1 - x)^{-i}$ by which to reduce them.

This serves to show how many terms of the resulting series are to be taken, for if l represent the rank of the last term, we must have $(l-1)(n+1) = < s$. Therefore the series is to be continued only so long as $s - (l-1)(n+1)$ is positive. As a numerical example, we may take the problem already solved in page 4, to find the chance of throwing 7 with two dice. Dice have no side marked with 0, therefore, before the formula can be applied to this and similar cases, each face or ball must be supposed to import one less than is marked on it, which amounts to substituting in the formula $s-n$ for s . We shall, however, not substitute in the formula so altered, but deduce its value in the particular case, as we have done in the general one.

The probability required will be the coefficient of x^7 in

$$\left\{ \frac{x^1 + x^2 + x^3 + x^4 + x^5 + x^6}{6} \right\}^2 = \left\{ \frac{x - x^7}{6(1-x)} \right\}^2 \\ = \frac{1}{36} (x^2 - 2x^3 + x^4) (1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \&c.)$$

which is $\frac{6}{36} = \frac{1}{6}$ as before.

If we want the probability of throwing 10 with three dice, we must find the coefficient of x^{10} in

$$\left\{ \frac{x^1 + x^2 + \dots + x^6}{6} \right\}^3 = \left\{ \frac{x - x^7}{6(1-x)} \right\}^3 \\ = \frac{1}{216} (x^3 - 3x^4 + \&c.) (1 + 3x + 6x^2 + \dots + 36x^5 + \&c.)$$

which is $\frac{36-9}{216} = \frac{1}{5}$, the required probability.

19. If we give to s the different values 0, 1, 2, ... s , successively, the sum of all the $s+1$ resulting series will of course be the probability that the sum of the numbers drawn will not be greater than s . In the last example, if we collect the coefficients of all the terms to x^{10} inclusive, we shall find the probability of not throwing more than 10 with three dice. It appears to be

$$\frac{1+3+6+10+15+21+28+36-3-9}{216} = \frac{108}{216} = \frac{1}{2}.$$

This probability is the foundation of the game of *Passe dix*.

20. If, instead of thus giving to s values which are increased by unity at each step, we suppose that s may have any value whatever between 0 and s , the probability must be estimated according to the rules of continuously varying quantities. The result in that case will be found identical with that given by our series, if in it we suppose s and n both infinite. The series then becomes

$$\frac{1}{1.2 \dots (i-1)n} \left\{ \left(\frac{s}{n} \right)^{i-1} - i \left(\frac{s}{n} - 1 \right)^{i-1} + \frac{i \cdot (i-1)}{1.2} \left(\frac{s}{n} - 2 \right)^{i-2} - \&c. \right\},$$

and the sum of all these series will be found by integrating this expression between the limits 0 and s , which gives

$$\frac{1}{1.2 \dots i} \left\{ \left(\frac{s}{n} \right)^i - i \left(\frac{s}{n} - 1 \right)^i + \frac{i \cdot (i-1)}{1.2} \left(\frac{s}{n} - 2 \right)^i - \&c. \right\},$$

the last or i^{th} term here being the last in which $\frac{s}{n} - (i-1)$ is positive.

This taken between the proper limits will give the probability that the sum of the numbers drawn is contained between those limits.

21. Laplace applies this equation, *Théorie Anal. des Prob.* p. 257, to finding the probability that the sum of the inclinations of the orbits of the planets would be contained within given limits, if all inclinations were similarly circumstanced.

There are 10 planets besides the Earth, namely, Mercury, Venus, Mars, Pallas, Juno, Ceres, Vesta, Jupiter, Saturn, and Uranus; therefore, $i = 10$ and the sum of their inclinations to that of the Earth at the beginning of 1801 was $82^{\circ} 16' 36''$, therefore

$$\frac{s}{n} = \frac{82 \cdot 27683}{90} = \cdot 914187.$$

Retaining only the first term of the preceding expression, because $\frac{s}{n} - 1$ is negative, the probability that the sum of the inclinations of the orbits of the planets would be comprised between the limits zero and $82^{\circ} \cdot 27683$, if all inclinations were similarly circumstanced, is

$$\frac{1}{1.2.3 \dots 10} \cdot (\cdot 914187)^{10} = \cdot 00000011235.$$

The question we have just solved is nearly the same as to determine the probability of the losses of an insurance company upon i policies upon persons of the same age being contained within certain limits.

22. A shilling is tossed into the air; B gives A a certain sum, in consideration of which A engages to pay B 2 pounds if the shilling falls head the first time; 4 pounds if it falls head the second time, and not before; 2^n pounds if it falls head the n^{th} time, and not before.

The expectation of B is

$$2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \dots + 2^n \times \frac{1}{2^n} = \text{£}n.$$

This is called the Petersburg problem, probably from the mention made of it by Daniel Bernoulli in the Transactions of the Petersburg Academy; it was first proposed by Montmort in the *Analyse des Jeux de Hasard*, and has been generally considered as involving a great paradox, because if it is agreed that the game shall not be discontinued till the shilling falls head, n must be made infinite, and the expectation of B is infinite; still no man of prudence would be disposed to venture even a small part of his fortune at this game. On account of the celebrity of this problem we have inserted it, but there is nothing paradoxical in the result more than in any other case of *long odds*: it shows, however, that in order to estimate the value of a contingent advantage to an individual, other elements must be considered, and that the moral expectation, as it is called, to distinguish it from the mathematical expectation, depends on a great many circumstances which it is very difficult to submit to calculation.

23. It is evident that the value of any sum is much greater to an individual who lives by his daily labour and is without any capital, than it is to an individual possessed of £100,000; we may, therefore, meet this difficulty by supposing that the value of a given sum to any individual, is proportional to and may be represented by that sum, divided by the whole of the fortune which he possesses.

Let a be the sum A is to pay B on the event of which the probability is p ,
 $b \dots \dots \dots B \dots \dots \dots A \dots \dots \dots q$,
 and let f be the fortune possessed by A. On the preceding hypothesis the expectation of A is

$$\frac{b q}{(f + b)} - \frac{a p}{(f - a)}.$$

If $b q = a p$, which must be the case in order that the wager may be *fair*, the expectation of A will be $> = < 0$ as

$$\frac{1}{f + b} - \frac{1}{f - a} > = < 0$$

$$f - a > = < f + b, \text{ or } a + b < = > 0;$$

a and b are both necessarily positive, therefore $a + b > 0$. It is, therefore, evident that according to this theory of the value of money, a wager under the most favourable circumstances consistent with honesty, injures the fortune of the gamester, because he loses more by losing than he can gain by gaining any wager, since the amount of it must be compared in the former case with his diminished, in the latter, with his augmented fortune.

24. Suppose the fortunes of A and B to be both equal to f , the expectation

$$\text{of A is } \frac{b q}{(f + b)} - \frac{a p}{(f - a)},$$

$$\text{of B is } \frac{a p}{(f + a)} - \frac{b q}{(f - b)}.$$

In order that the expectations of A and B may be equal

$$\frac{b q}{(f+b)} - \frac{a p}{(f-a)} = \frac{a p}{(f+a)} - \frac{b q}{(f-b)}$$

$$\frac{2 f a p}{f^2 - a^2} = \frac{2 f b q}{f^2 - b^2} \quad \frac{a}{b} = \frac{q}{p} \cdot \frac{f^2 - a^2}{f^2 - b^2}$$

$\frac{a}{b} > = < \frac{q}{p}$ as $f^2 - a^2 > = < f^2 - b^2$, as $b > = < a$; therefore if $a > b$, that is, if $p < q$, (a and b are the odds) a is not so much greater than b , as it would be if the moral expectation were not taken into account. In practice we believe this is always the case, as the reader may have observed in the example we took, page 9.

Ex. 14. Let p be the probability of a vessel coming into port, q the probability of a total loss, a the value of that part of the cargo in any vessel, which belongs to A, let there be n vessels, and f , as before, A's fortune. The mathematical value of A's expectation, the $n a$ goods being equally distributed over n vessels, is

$$n a p^n + (n-1) a n p^{n-1} q + (n-2) a \cdot \frac{n(n-1)}{1 \cdot 2} p^{n-2} q^2 + \&c.$$

$$= n a p \{p + q\}^{n-1} = n a p, \text{ since } p + q = 1,$$

which is the same as if they were all on board the same vessel; but, on the preceding hypothesis, the value of A's expectation is

$$n a \frac{p^n}{f + n a} + (n-1) a \cdot \frac{n p^{n-1} q}{f + (n-1) a} + (n-2) a \cdot \frac{n \cdot (n-1)}{1 \cdot 2} \frac{p^{n-2} q^2}{f + (n-2) a} + \&c.$$

Where every denominator but the first is less than $(f + n a)$, and, therefore, this series is greater than the sum of the numerators divided by $(f + n a)$,

$$\text{that is, greater than } \frac{n a p \cdot (p+q)^{n-1}}{f + n a}, \text{ that is, than } \frac{n a p}{f + n a}.$$

The value of A's expectation is, therefore, greater than $\frac{n a p}{f + n a}$, which is

the value of his expectation when all his goods are on board the same vessel.

This principle of the distribution of risk is well known to every merchant, and, in fact, affords him all the advantages derived from insurance. It is on this principle that the East India Company never insure their vessels.

The hypothesis we have adopted in the preceding problems was suggested by the celebrated naturalist, Buffon.

25. Daniel Bernoulli supposes that the value of the fortune of any individual is made up of an infinite number of indefinitely small elements, the value of each of which is inversely as the capital already formed, so that if ϕ represent a small element of the fortune f , and $f_1, f_2, \&c.$, its successive amounts, the value of the whole fortune f will be

$$\frac{k \phi}{f_1} + \frac{k \phi}{f_2} + \dots + \frac{k \phi}{f - \phi} \quad k \text{ being some constant quantity.}$$

When ϕ is indefinitely diminished, this series $= k \log. \left(\frac{f}{f_1} \right)$. The divisor f_1 denotes that part of the fortune of the individual which is absolutely unalienable, and below which his fortune cannot sink. This theory affords us

solutions to the problem preceding, and similar problems analogous to those obtained on the supposition of Buffon.

We shall not dwell longer on this subject, 'as these hypotheses, although they may serve, in some measure, to show the difference which exists between the mathematical value of any sum and its value in practice, are quite arbitrary.

26. The search after a method of enabling a gamester to win with certainty from his antagonist, who has a greater probability in his favour, has wasted as much ingenuity as the attempt to discover perpetual motion. At all gaming tables an advantage is given by the laws of the game to the banker; and many infatuated persons, in the vain hope of detecting some scheme for rendering that advantage nugatory, have spent years in registering the course of the play with a degree of patient industry which, exerted in another direction, might have made them useful and distinguished members of society. One favourite scheme is so celebrated as to have acquired a particular name; it is called the *Martingale*, or *Double or Quits*, and consists in doubling the last stake after every loss. In order that this may be permanently successful, the player requires not only an immense capital, but an unlimited permission of staking. It is not very easy to show mathematically the amount of the player's expectation who uses the *martingale*, on account of the various order in which the gains and losses may be supposed to follow each other; instead of attempting it, we shall give an analysis of another scheme, in which the same difficulty does not occur. This consists in increasing the stake by a fixed sum after every loss, and diminishing it by the same after every gain. The inventors of this mode of betting looked upon it as infallible, and indeed there is something in it which might easily deceive the unwary; for it can be shown, that if the number of games won and lost be the same, no matter in what order this takes place, the result is always a gain to the player who bets upon this principle. Notwithstanding this specious circumstance, we shall show that the value of the player's expectation of gain, when his probability of winning a single game exceeds $\frac{1}{2}$, is never so great as his expectation of loss when this probability falls short of $\frac{1}{2}$ by the same quantity.

27. Let a be the original stake, b the quantity by which it is increased or diminished after every loss or gain, $m + n$ the whole number of games played. The first thing to determine is the player's gain who wins m and loses n games.

His first stake $= a$, therefore his first gain $= \pm a$, and may be represented generally by $a(-1)^a$ which becomes $\pm a$ as a is even or odd. If he wins, i. e. if a be even, his second stake is $a - b$, and it is $a + b$ if he loses, i. e. if a be odd: therefore his second stake generally is $a - b(-1)^a$ and his second gain may be represented in the same general manner by $\{a - b(-1)^a\}(-1)^b$. For the same reason his third stake will be $a - b(-1)^a - b(-1)^b$, and therefore his third gain is $\{a - b((-1)^a + (-1)^b)\}(-1)^c$, and so on.

We have, therefore, the following table of gains:

1st gain	$= a(-1)^a,$
2d ..	$= a(-1)^b - b\{(-1)^a\}(-1)^b,$
3d ..	$= a(-1)^c - b\{(-1)^a + (-1)^b\}(-1)^c,$
4th ..	$= a(-1)^d - b\{(-1)^a + (-1)^b + (-1)^c\}(-1)^d,$
&c.	&c.

There are $m + n$ stakes, and we have determined nothing of the quantities $a, b, c, \&c.$ except that m are even, and n odd, because by supposition the player wins m games, and loses n . Therefore m of the quantities $(-1)^a,$

$(-1)^a$, &c. will each $= +1$, and the remaining n each $= -1$. The coefficient of a in the sum of the gains is the sum of all these quantities, and therefore $= m - n$. The coefficient of b is the sum of the products of them all two by two, and therefore is equal to the coefficient of the third term of an equation which has m roots $= +1$, and n roots $= -1$, i. e. of $(x-1)^m (x+1)^n$, i. e.

$$= \frac{m \cdot (m-1)}{1 \cdot 2} - mn + \frac{n \cdot (n-1)}{1 \cdot 2} = \frac{(m-n)^2 - (m+n)}{2}.$$

Therefore, the player's gain $= (m-n) a + \frac{(m+n) - (m-n)^2}{2} b$; if

$m = n$ this reduces itself to $\frac{(m+n)b}{2}$, and it is plain that these results are

quite independent of the order in which the gains and losses follow each other. This very elegant solution was given by Mr. Babbage, in the *Edinburgh Transactions* for 1821; it remains now to estimate the player's expectation, whose probability of winning any single game $= p$. Let $m+n = i$.

The player's gain, if he wins m games, $= (m-n) a + \frac{m+n - (m-n)^2}{1 \cdot 2} b$

$= \left\{ i a + \frac{i(i-1)}{1 \cdot 2} b \right\} + 2m \{ a + (i-1)b \} - 2m \cdot (m-1)b$, since

$n = i - m$; and in order to get the player's expectation we must multiply this into the term of $\{ p + (1-p) \}^i$ of which the argument is $p^n \cdot (1-p)^{i-n}$, and take the sum of all those products; giving m , which is now the only variable, every value from 0 to i both inclusive. This product is

$$= \frac{i \cdot (i-1) \dots 1}{1 \cdot 2 \dots m \cdot 1 \cdot 2 \dots n} p^m (1-p)^{i-m}$$

$$\times \left\{ \left\{ i a + \frac{i(i-1)}{1 \cdot 2} b \right\} - 2m \{ a + (i-1)b \} + 2m(m-1)b \right\}.$$

Therefore the sum of all the values of this product is

$$\begin{aligned} & \left\{ i a + \frac{i(i-1)}{1 \cdot 2} b \right\} \times \text{the sum of all } \frac{i \cdot (i-1) \dots 1}{1 \cdot 2 \dots m \cdot 1 \cdot 2 \dots n} p^m (1-p)^{i-m} \\ & + 2 \cdot \{ a + (i-1)b \} i p \times \dots \dots \dots \frac{(i-1) \cdot (i-2) \dots 1}{1 \cdot 2 \dots (m-1) \cdot 1 \cdot 2 \dots n} p^{m-1} (1-p)^{i-m} \\ & - 2b \cdot i \cdot (i-1) p^2 \times \dots \dots \dots \frac{(i-2) \cdot (i-3) \dots 1}{1 \cdot 2 \dots (m-2) \cdot 1 \cdot 2 \dots n} p^{m-2} (1-p)^{i-m}. \end{aligned}$$

When every value from 0 to i inclusive is given to m , the sums of all the values of these three right-hand factors, rejecting those in which the index of p is negative, severally become $\{ p + (1-p) \}^i$, $\{ p + (1-p) \}^{i-1}$, and $\{ p + (1-p) \}^{i-2}$, all equal to unity.

Therefore the sum of all the values of the products

$$\begin{aligned} & = - \left\{ i a + \frac{i(i-1)}{1 \cdot 2} b \right\} + 2 i p \cdot \{ a + (i-1)b \} - 2 i (i-1) p^2 b \\ & = i \cdot (2p-1) \cdot a - \frac{i \cdot (i-1)}{1 \cdot 2} (2p-1)^2 b. \end{aligned}$$

If $p = \frac{1}{2} + x$, this expectation of gain $= 2 i x a - 2 i (i-1) x^2 b$, and

If $p = \frac{1}{2} - x$ the expectation of loss $= 2 i x a + 2 i (i-1) x^2 b$. The expectation of a player, who is entirely ignorant of the value of p , is found by integrating the expression here found

$$\int \left\{ i a \cdot (2p-1) - \frac{i(i-1)}{1 \cdot 2} b \cdot (2p-1)^2 \right\} dp \\ = \frac{3 i a \cdot (2p-1)^3 - i \cdot (i-1) \cdot b \cdot (2p-1)^3}{12}$$

and from $p = 0$ to $p = 1$, $= -\frac{i \cdot (i-1) \cdot b}{6}$.

It should be observed that this solution only applies to the case when there is a limiting equation between a and b , such that $a - (i-1)b > 0$, otherwise there might be a conjunction in the game, in which the player could not follow the rules of this scheme, and consequently would alter his expectation. If this be not attended to, the theorem supposes, what can never take place in practice, that the player has the power of reducing his stake below zero, that is, of taking his adversary's situation in some point of the game.

28. Let there be two conflicting events P and Q, of which the probabilities are p and q respectively, and let $m+n$ trials take place; the probability that the event P will happen m times, and the event Q, n times, without regard to the order in which they succeed each other, is

$$\frac{(m+n) \cdot (m+n-1) \dots 2 \cdot 1}{1 \cdot 2 \dots m \cdot 1 \cdot 2 \dots n} p^m \cdot q^n,$$

which we shall represent by p_m . Similarly, the probability that the event P will happen $m+1$ times and the event Q, $(n-1)$ times, is

$$\frac{(m+n) \cdot (m+n-1) \dots 2 \cdot 1}{1 \cdot 2 \dots (m+1) 1 \cdot 2 \dots (n-1)} p^{m+1} \cdot q^{n-1} = p_{m+1}.$$

If $p_{m+1} > p_m$, $\frac{p_{m+1}}{p_m} > 1$; $p > \frac{(m+1) \cdot q}{n}$; and since $p + q = 1$,

$m > (m+n)p - q$, and this continues until $m = (m+n)p - q$, in which case $p_m = p_{m+1}$; and, since m must be a whole number, the greatest term in the development of $(p+q)^{m+n}$ is that when $m = (m+n)p - q$, if $(m+n)p - q$ be a whole number, or, if not a whole number, the next greater.

Suppose, for instance, $p = \frac{2}{3}$ and consequently $q = \frac{1}{3}$, and let $m+n = 17$, m is $\frac{17 \times 2 - 1}{3}$, or $\frac{33}{3} = 11$; so that the most probable event, as compared with any other event which can occur in 17 trials, is a repetition of P 11 times, and Q 6 times.

If $m+n = 18$, m is the whole number next greater than $\frac{18 \times 2 - 1}{3}$, $= \frac{35}{3}$; which is 12, and the most probable event as compared with any other which can happen in 18 trials, is a repetition of P 12 times, and of Q 6 times.

When $m+n = \frac{w}{p}$, w being a whole number; since m is the next whole number greater than $(m+n)p - q$, and since $(m+n)p = w$, and q is necessarily a proper fraction, $(m+n)p$ or w is the next whole number greater than $(m+n)p - q$, and, therefore, $n = w = (m+n)p$, $n = (m+n)(1-p) = (m+n)q$; $\frac{m}{n} = \frac{p}{q}$, that is to say, the event most likely to happen is a combination in which the number of repetitions of p and q is proportional to the simple probability of the happening of each.

29. It is only as compared with any other single combination, that the one we have just mentioned increases in probability with the number of trials: if we estimate the abstract probability of the event corresponding with this *maximum* term, we shall easily find that it diminishes as the number of trials increases.

For this purpose let the maximum term in $m+n$ trials be represented by p_m , we have already seen that

$$p_m = \frac{(m+n) \cdot (m+n-1) \dots 2 \cdot 1}{1 \cdot 2 \dots m \dots 1 \cdot 2 \dots n} p^m q^n,$$

where $m > (m+n)p - q$, that is $> (m+n+1)p - 1$,
and $< (m+n+1)p$.

In one more trial

$$p_{m'} = \frac{(m+n+1) \cdot (m+n) \dots 2 \cdot 1}{1 \cdot 2 \dots m' \dots 1 \cdot 2 \dots n'} p^{m'} q^{n'},$$

m' being limited in the same manner,

$$\begin{aligned} m' &> (m+n+2)p - 1, \text{ that is } > m+p-1, \\ &< (m+n+2)p, \text{ that is } < m+p+1, \\ \therefore m' &\text{ is either } m \text{ or } m+1. \end{aligned}$$

$$\text{If } m' = m, \quad \frac{p_{m'}}{p_m} = \frac{m+n+1}{n+1} q = \frac{n+1+m-(m+n+1)p}{n+1} < 1.$$

$$\text{If } m' = m+1, \quad \frac{p_{m'}}{p_m} = \frac{m+n+1}{m+1} \cdot p < 1,$$

$\therefore$ in both cases, which will go on occurring successively, $p_{m'} < p_m$.

30. There is, however, another probability connected with the most probable combination, which does increase continually with the number of trials, for it is always possible to assign a number of trials, such as to give any required probability that the difference between the ratio of the number of repetitions of the events, and the simple probabilities of the events, shall lie within any given limits. Thus, if there are 3 white balls in a bag, and 2 black balls, we can always assign a number of trials such as to give any probability as near as we please to certainty, that the difference between $\frac{3}{5}$ and the ratio of the number of white balls drawn to the number of black balls, shall lie between given limits, however near those limits may be assumed. This is a theorem of the highest importance in the theory of probability, and indeed, it is on the converse of it that the value of experience depends. We shall endeavour to prove it as shortly as possible, and it will facilitate this object if, instead of representing the simple probabilities by p and q , as we have

hitherto done, we express them by the two fractions $\frac{a}{a+b}$ and $\frac{b}{a+b}$, in which a and b are both whole numbers.

31. The events of which the probabilities are $\frac{a}{a+b}$ $\frac{b}{a+b}$ would be

repeated in $m(a+b)$ trials exactly ma and mb times, if they were combined in the ratio of their simple probabilities. Let us suppose that the number of times that the first will be repeated in the observed event lies between the limits $ma-m$ and $ma+m$, that is to say, that there will not be fewer than $ma-m$ and not more than $ma+m$ recurrences of that event in the $m(a+b)$ trials. The probability that this will be the case is

the sum of the $2m+1$ terms in the development of $\left(\frac{a}{a+b} + \frac{b}{a+b}\right)^{m(s+1)}$

from the term whose argument is α^{m-n} to the term whose argument is α^{m+n} , both inclusive. We shall call these two last-mentioned terms the first and second limiting terms, and it is clear that the maximum term, the argument of which is α^m , lies between them.

32. The whole series may thus be written out at length:

$$\left(\frac{a}{a+b} + \frac{b}{a+b}\right)^{n(a+b)} = p_{n(a+b)} + p_{n(a+b)-1} + \dots + p_{n(a+b)-n}$$

First limiting term.

Maximum.

Second limiting term.

$$\left| \begin{array}{c} + p_{m+m} \dots + p_{m+s} \\ \text{Parcel of the first limit.} \end{array} \right| + p_m \left| \begin{array}{c} + p_{m-1} \dots + p_{m-n} \\ \text{Parcel of the second limit.} \end{array} \right|$$

$$+ p_{m-1} \dots + p_1 + p_0$$

the index at the foot of p always denoting the power to which a is involved in that term. Our object is to show that the $2m+1$ terms within the limits, may be made as many times greater than the rest of the series as we please; and we shall do this by showing that the first m of these $2m+1$ terms, which we will call the parcel of the first limit, can be made as many times greater as we please than all which precede them, and the last m terms, or the parcel of the second limit, as many times greater as we please than all which follow them.

33. There are $m b - m$ terms which precede our first limit, which may be classed in $(b - 1)$ parcels, each containing m successive terms, and similarly the $m a - m$ terms, after the second limit, may be classed in $(a - 1)$ parcels, each containing m successive terms. As the maximum term p_{n_0} is in the middle of our limits, and as the values of the terms increase from each end of the series up to the maximum term, the sum of all the $(b - 1)$ parcels before the first limit will be less than $(b - 1)$ times the parcel next before the first limit; and the sum of all the $(a - 1)$ parcels beyond the second limit will be less than $(a - 1)$ times the parcel next following the second limit. It is also plain, for the same reason, that the parcel of the first limit is greater than the parcel which immediately precedes the first limit, and, since the ratio of the maximum term to the first limiting term, or the ratio of p_{n_0} to p_{n_0+m} , is less than the ratio of any term in the parcel of the first limit, to the corresponding term in the parcel next preceding the first limit, (because these ratios continually approach nearer to a ratio of equality as they approach the maximum term,) it follows that the ratio of p_{n_0} to p_{n_0+m} is less than the ratio of the whole parcel of the first limit to the whole parcel immediately preceding it.

34. If, therefore, we can show that $p_{n,i}$ can, by a proper assumption of m , be made greater than $i.(b-1)$ times $p_{n+i,m}$ however great i is taken, it will follow that this value of m will make the parcel of the first limit still

greater than $i.(b-1)$ times the parcel immediately preceding it, and very much greater than i times the sum of all the parcels which precede it. Exactly the same reasoning will show that if p_{m-1} can be made greater than

$i.(a-1)$ times p_{m-2} , that is, $\frac{p_{m-1}}{p_{m-2}} > i.(a-1)$ the parcel of the second

limit, will, with the same value of m , be more than i times greater than all the parcels which follow it.

35. Let a be that one of the quantities a and b , which is not the least. A value of m , which will make p_{m-1} greater than $i.(a-1)$ times p_{m-2} , and than $i.(a-1)$ times p_{m-3} , will evidently satisfy both the above-mentioned conditions.

$$\frac{p_{m-1}}{p_{m-2}} = \frac{(ma+1) \dots (ma+m)}{(mb - \{m-1\}) \dots mb} \left(\frac{b}{a}\right)^m = \frac{(mab+b) \dots (mab+mb)}{(mab - \{m-1\}a) \dots mab}.$$

The last pair of factors here is $\frac{mab+mb}{mab}$, or $\frac{a+1}{a}$, and any other pair, such as the r^{th} , is

$$\frac{mab+rb}{mab - (m-r)a}, \text{ which is } > \frac{a+1}{a},$$

if

$$mab+rb > (a+1)(mb - m + r),$$

that is, if

$$b < a+1,$$

which by supposition it is. Therefore

$$\frac{p_{m-1}}{p_{m-2}} > \left(\frac{a+1}{a}\right)^m,$$

for there are m factors in this continued product.

$$\frac{p_{m-1}}{p_{m-3}} = \frac{(mb+1) \dots (mb+m)}{(ma - \{m-1\}) \dots ma} \left(\frac{a}{b}\right)^m = \frac{(mab+a) \dots (mab+ma)}{(mab - \{m-1\}b) \dots mab}.$$

Any factor, as the r^{th} , here

$$= \frac{mab+ra}{mab - (m-r)b}, \text{ which is } > \frac{a+1}{a},$$

if

$$a^2(mb+r) > (a+1).b(ma - \{m-r\}),$$

that is, if

$$ra(a-b) + (m-r).b > 0,$$

which it must always be, for neither $a-b$, nor $m-r$, can ever become negative.

$$\therefore \frac{p_{m-1}}{p_{m-3}} > \left(\frac{a+1}{a}\right)^m. \text{ Assume } \left(\frac{a+1}{a}\right)^m = i.(a-1),$$

or

$$m = \frac{\log. i + \log. (a-1)}{\log. (a+1) - \log. a},$$

and with this value of m both the necessary conditions are satisfied.

36. Therefore the probability

$$= \frac{\text{1st parcel} + 2\text{d parcel} + a_m}{\text{1st parcel} + 2\text{d parcel} + a_m + \text{sum of other parcels}},$$

or since $\text{1st parcel} + 2\text{d parcel} > i.(\text{sum of other parcels})$

$$= i (\text{sum of other parcels}) + k,$$

our probability

$$= \frac{i \cdot (\text{sum of other parcels}) + k + a_m}{(i+1) (\text{sum of other parcels}) + k + a_m} > \frac{i}{i+1},$$

since every proper fraction is diminished by taking the same quantity (in this instance $k + a_m$) from its numerator and denominator.

37. We thus have a probability, $> \frac{i}{i+1}$, that the number of recurrences

of the event, whose probability is $\frac{a}{a+b}$, will, in $m \cdot (a+b)$ trials, lie between

the numbers $(ma+m)$, $(ma-m)$, however great i be taken, and the limits of the ratio of the number of repetitions of this event to the number of trials are $\frac{ma+m}{m \cdot (a+b)}$ and $\frac{ma-m}{m \cdot (a+b)}$, or $\frac{a+1}{a+b}$ and $\frac{a-1}{a+b}$, the difference of which is

$\frac{2}{a+b}$; and since the only restriction on the value of a and b is, that they

must be in the proportion of p to q , we may increase $a+b$ at pleasure, and thus bring the limits of this ratio as close as we please, and yet have a proba-

bility, $> \frac{i}{i+1}$, that the observed ratio will be between them. If all that is

required is that there will not be fewer than $ma-m$ recurrences, the probability clearly becomes much greater, as also the probability that there will not be more than $ma+m$.

It only remains to give an example to show fully how this theorem is applied.

Ex. 15. A bag contains three white balls and two black balls, from which a ball is repeatedly drawn and replaced. Required the number of trials in which the odds will be at least 1000 to 1 that the ratio of the number of times that a white ball is drawn to the whole number of drawings is not less

than $\frac{29}{50}$, and not greater than $\frac{31}{50}$.

The value of i is here 1000; $a = 30$, and $b = 20$.

$$\therefore m = \frac{\log. i + \log. (a-1)}{\log. (a+1) - \log. a} = \frac{\log. 1000 + \log. 29}{\log. 31 - \log. 30}$$

$$= \frac{4.462398}{0.0142404} > 313 < 314.$$

Therefore we must take $m = 314$, and the number of trials $= m(a+b) = 15700$, in which number of drawings the odds will be more than 1000 to 1 that a white ball will have been drawn not less than $ma-m$, or 9106 times, and not more than $ma+m$, or 9734 times.

If the odds had been required to be at least 10,000 to 1, or 100,000 to 1, we need only change the value of i , which gives, in the first case,

$$m = \frac{5.462398}{0.0142404} < 384 \text{ and } m(a+b) = 19200;$$

and in the second case,

$$m = \frac{6.462398}{0.0142404} < 454 \text{ and } m(a+b) = 22700,$$

which serves to show how much more rapidly the probability increases than the number of trials; for after the first 15,700 trials, the addition of 3500 new trials increases the odds tenfold, and 3500 more increases them tenfold again.

38. This highly important theorem is due to James Bernoulli, a celebrated mathematician of the last century, whose name it bears.

39. We have supposed known the number of favourable and unfavourable cases similarly circumstanced in the problems we have hitherto considered. Thus, in the problems relating to dice, we took for granted the form of the dice, and also their homogeneity.

40. However nearly any die may fulfil these given conditions, it will not do so strictly; and if we investigate the probability of any throw upon the principles hitherto developed, we obtain a result approximately correct, and of which the error depends on the inaccuracy of our hypothesis. The knowledge that a defect of homogeneity is possible, renders the return of the same face in several repeated trials more probable than it would be, if the die were known to be homogeneous. In tossing up a shilling, the probability of its falling heads, or the reverse, twice successively, is rather greater than $\frac{1}{4}$.

41. If such considerations apply to these very simple questions, it will readily be seen how difficult it is to estimate mathematically those probabilities which depend on more complicated circumstances; as the probability of an individual living a given number of years, or the probability of the truth of any assertion. Truths of definition are the only certain propositions. We shall not stop to inquire whether any limit separates truths of definition from propositions which rest upon experience; the distinction however may be admitted.

42. It is impossible to suppose that "*a part is greater than the whole*" without involving a contradiction to the sense in which the words forming this sentence are understood, so that the truth of the proposition, that "*a part is less than the whole*" results from the very definition of the words which compose this sentence. If on the other hand we consider the proposition that the sun will rise to-morrow; the number of times we believe the sun to have risen daily without interruption induces us to believe that the sun will rise again. Most of our opinions arise from our experience of the past, and rest upon probabilities of this kind.

43. In order to obtain mathematical solutions of problems similar to these we must revert to games of chance. Any problem in chances may be represented by throws with dice of different forms, or by drawings from bags containing balls of different colours. Nor is it any objection to the results we obtain by this means, that no dice can be formed which exactly fulfil the conditions we suppose them to do, any more than it is to the theorems in Euclid that the lines which compose the diagrams are not mathematically straight.

44. When our knowledge of the number of cases similarly circumstanced is imperfect, the probability of an event is still deduced upon the same principles as those hitherto developed. We have recourse to *hypotheses*, and having estimated their probability, the probability of any future event which depends on them, is easily deduced.

45. The probability of each hypothesis by definition is the number of cases which favour this hypothesis divided by the whole number of cases possible.

Ex. 16. Let us suppose a bag to contain three balls, and that we are uncertain whether of these balls two are black and one is white, or one is black and two are white, and that a white ball has been drawn.

Let us call these balls 1, 2, 3, and let us also suppose that the uncertainty

is with respect to the colour of No. 2.

	No. 1.	No. 2.	No. 3.
1st hypothesis,	black,	black,	white.
2nd ..	black,	white,	white.

On the first hypothesis No. 3 must have been drawn; we have, therefore, only one case which favours this hypothesis. On the second hypothesis, either No. 2 or No. 3 has been drawn, so that we have two cases which favour this hypothesis, and, therefore, the probabilities of these hypotheses respectively are $\frac{1}{1+2}$ and $\frac{2}{1+2}$, or $\frac{1}{3}$ and $\frac{2}{3}$.

In order to extend the principle of this reasoning to the general case, let us suppose that an event has been observed which must have resulted from one of a given number of causes. Let the probability of the existence of one of these causes have been estimated at P before the observed event took place, and let the probability of the observed event calculated upon the certitude of that cause be called p .

The probability that the event will happen in consequence of that cause $= Pp$, and the probability that the event will happen, without reference to any particular cause $= \sum . Pp$; extending the sign of summation to all the possible causes.

The probability that the event will happen in consequence of the selected cause, (or Pp), may be estimated in a different manner: it equals the product of the probability that it will happen, (or $\Sigma.Pp$), by the probability that if it does happen, it will be in consequence of that cause: the latter is evidently the probability of the selected cause derived from the observation.

Therefore the derived probability of that cause = $\frac{Pp}{\Sigma.Pp}$, which result may be stated in the following important theorem.

The probability of any hypothesis is the probability of the observed event upon this hypothesis multiplied by the probability of the hypothesis antecedently to the observation divided by the sum of the products which are formed in the same manner from all the hypotheses.

46. The probability antecedent to the observations under consideration is called the *a priori* probability; but in using this term it must be remembered that it is relative to a given epoch.

Ex. 17. Thus in the instance of a bag containing three balls, of which a white one has been once drawn and replaced. There are three possible suppositions: first, all three are white; second, two only are white; third, only one is white. On the first supposition the probability of the event observed is certainty or 1, on the second the probability is $\frac{2}{3}$, on the third it is $\frac{1}{3}$.

Therefore the probability of the first hypothesis = $\frac{1}{1 + \frac{2}{3} + \frac{1}{3}} = \frac{1}{2}$.

$$\text{second} = \frac{\frac{2}{3}}{1 + \frac{2}{3} + \frac{1}{3}} = \frac{1}{3},$$

" " third " $= \frac{\frac{1}{3}}{1 + \frac{2}{3} + \frac{1}{3}} = \frac{1}{6}$

It is worth observing that these conclusions would not be affected by retaining all the four hypotheses which might have been made before the observation. For the probability of the observed event on the hypothesis we have rejected, namely, that all three balls are white, being $= 0$, the probability of the others will not be altered by including it also in the sum of probabilities which make up the denominator of the above-written fractions.

47. Let the ratio of the white balls to the whole number of balls be any of the following quantities,

$$x \dots\dots 2x \dots\dots 3x \dots\dots ix,$$

let any of these hypotheses be equally probable *a priori*, and let a white ball have been drawn.

The probability of the event observed, namely, the drawing a white ball, on the first hypothesis is x , the *a priori* probability of this hypothesis is $\frac{1}{i}$, because there are i hypotheses equally probable; therefore the probability of this hypothesis is

$$\frac{1}{i} \times \frac{x}{1 + 2 + 3 \dots i} = \frac{2}{i(i+1)}.$$

Similarly, the probability of the second hypothesis is $\frac{2 \times 2}{i(i+1)}$, of the third is $\frac{3 \times 2}{i(i+1)}$, and so on.

The probability of drawing a white ball in a future trial, after replacing the ball drawn, if the first hypothesis were the true one, would be x ; the probability of this hypothesis is $\frac{2}{i(i+1)}$; therefore, the probability of drawing a white ball, considering only this hypothesis, is $\frac{2x}{i(i+1)}$.

Similarly, considering the second hypothesis, the probability is $\frac{2x \times 2}{i(i+1)}$,

“ third “ $\frac{2x \times 3}{i(i+1)}$,

and the sum of all these, or the probability of drawing a white ball again, is

$$\frac{2x}{i(i+1)} \{ 1 + 2 + 3 + \dots + i \}.$$

In order to find the sum of this series, we shall employ a method of greater generality than is necessary in this particular case, because it furnishes the readiest method of summing a great many series of the same kind.

$$e^x = 1 + x + \frac{x^2}{1.2} + \&c.$$

$$e^{2x} = 1 + 2x + \frac{2^2 \times x^2}{1.2} + \&c.$$

$$e^x = 1 + ix + \frac{i^2 x^2}{1 \cdot 2} + \&c.$$

$\therefore 1 + 2^i + 3^i \dots + i^i$ is the coefficient of $\frac{x^i}{i}$ in $e^x + e^{2x} \dots e^{ix}$,

$$\begin{aligned} \text{" " " } & \text{or } \frac{e^x (e^x - 1)}{e^x - 1} \text{ or } \frac{e^x - 1}{1 - e^{-x}}, \\ \text{" " " } & \text{or } i + \frac{i^2 x}{2} + \frac{i^3 x^2}{6} + \&c. \\ & \frac{1 - \frac{x}{2} + \frac{x^2}{6} - \&c.} \end{aligned}$$

If we effect the division of $\frac{1}{1 - \frac{x}{2} + \frac{x^2}{6} - \&c.}$ the three first terms will be

found to be $1 + \frac{x}{2} + \frac{x^2}{12}$, and all beyond involve higher powers of x than the square, and therefore need not be considered. Multiplying $i + \frac{i^2 x}{2} + \frac{i^3 x^2}{6}$ by $1 + \frac{x}{2} + \frac{x^2}{1 \cdot 2}$, we get for the coefficient of $\frac{x^i}{2}$, $\frac{2i^3 + 3i^2 + i}{6}$, therefore

$$1 + 2^i \dots + i^i = \frac{2i^3 + 3i^2 + i}{6} = \frac{i(i+1)(2i+1)}{6},$$

and the probability in question is

$$\frac{2x}{i \cdot i + 1} \left\{ \frac{i(i+1)(2i+1)}{6} \right\} = \frac{(2i+1) \cdot x}{3}.$$

When i is very great, this fraction approximates to $\frac{2ix}{3}$; if, therefore,

the ratio of the white balls may be any ratio between 0 and unity, that is, if we have no data to determine that some of these values are more probable than others, $ix = 1$, and this probability is $\frac{2}{3}$.

48. Let the ratio of the white balls to the whole number of balls, be any of the following, $\Delta x, 2 \Delta x, 3 \Delta x \dots i \Delta x$, and consequently the ratio of the black balls to the whole number of balls,

$$1 - \Delta x, 1 - 2 \Delta x, 1 - 3 \Delta x \dots 1 - i \Delta x,$$

and let m white balls have been drawn, and n black, in any given order.

The probability of the event observed on the hypothesis that $i \Delta x$ is the ratio of the white balls to the whole number of balls is

$$\frac{(m+n) \cdot (m+n-1) \dots 1}{1 \cdot 2 \cdot 3 \dots m \cdot 1 \cdot 2 \cdot 3 \dots n} \times (i \Delta x)^m (1 - i \Delta x)^n,$$

and the probability of this hypothesis is

$$\frac{(i \Delta x)^m (1 - i \Delta x)^n}{\Delta x^m (1 - \Delta x)^n + (2 \Delta x)^m (1 - 2 \Delta x)^n + \&c.}$$

The probability of drawing m' white balls, and n' black balls, in $m' + n'$ future trials upon this hypothesis, is

$$\frac{(m' + n') (m' + n' - 1) \dots \dots 1}{1 \cdot 2 \cdot 3 \dots m' \cdot 1 \cdot 2 \cdot 3 \dots n'} (i \Delta x)^{m'} (1 - i \Delta x)^{n'},$$

multiplying this by the probability of the hypothesis, the probability of drawing m' white balls and n' black balls, considering only this hypothesis, is

$$\frac{(m' + n') (m' + n' - 1) \dots \dots 1}{1 \cdot 2 \cdot 3 \dots m' \cdot 1 \cdot 2 \cdot 3 \dots n'} \\ \times \frac{(i \Delta x)^{m' + n'} (1 - i \Delta x)^{n + m'}}{\Delta x^n (1 - \Delta x)^n + (2 \Delta x)^n (1 - 2 \Delta x)^n + \&c.},$$

and the probability of drawing m' white balls and n' black balls, considering all the hypotheses, is

$$\frac{(m' + n') (m' + n' - 1) \dots \dots 1}{1 \cdot 2 \cdot 3 \dots m' \cdot 1 \cdot 2 \cdot 3 \dots n'} \\ \times \left\{ \frac{\Delta x^{n + m'} (1 - \Delta x)^{n + m'} + (2 \Delta x)^{n + m'} (1 - 2 \Delta x)^{n + m'} + \&c.}{\Delta x^n (1 - \Delta x)^n + (2 \Delta x)^n (1 - 2 \Delta x)^n + \&c.} \right\}.$$

Let Δx be infinitely diminished, and let $i \Delta x = 1$, that is, let any ratio of white balls to the whole number of balls contained in the bag to Δx be equally possible between zero and unity, then this expression becomes

$$\frac{(m' + n') (m' + n' - 1) \dots \dots 1}{1 \cdot 2 \cdot 3 \dots m' \cdot 1 \cdot 2 \cdot 3 \dots n'} \frac{\int x^{n + m'} (1 - x)^{n + m'} dx}{\int x^n (1 - x)^n dx},$$

taken between the limits $x = 0$ and $x = 1$.

$$\int x^n (1 - x)^n dx = \frac{x^{n+1}}{n+1} (1 - x)^n + \frac{n}{n+1} \int x^{n+1} (1 - x)^{n-1} dx \\ = \frac{x^{n+1}}{n+1} (1 - x)^n + \frac{n}{(n+1)(n+2)} x^{n+2} (1 - x)^{n-1} \\ + \frac{n \cdot (n-1)}{(n+1)(n+2)} \int x^{n+2} (1 - x)^{n-2} dx \\ = \frac{x^{n+1}}{n+1} (1 - x)^n + \frac{n}{(n+1)(n+2)} x^{n+2} (1 - x)^{n-1} \\ + \frac{n \cdot (n-1)}{(n+1)(n+2)(n+3)} x^{n+3} (1 - x)^{n-2} \\ \dots \dots + \frac{n \cdot (n-1) \cdot (n-2) \dots \dots 1}{(n+1)(n+2) \dots (n+n+1)} x^{n+n+1}.$$

When $x = 0$, all these terms vanish; when $x = 1$, all vanish except the last; therefore the integral required, taken between the limits $x = 0$ and $x = 1$, is

$$\frac{n \cdot (n-1) \cdot (n-2) \dots \dots 1}{(n+1)(n+2)(n+3) \dots (n+n+1)} \\ \int x^{n+n} (1 - x)^{n+n} dx, \text{ taken between the same limits, is} \\ \frac{(n + n') (n + n' - 1) (n + n' - 2) \dots \dots 1}{(m + m' + 1) (m + m' + 2) \dots (m + m' + n + n' + 1)},$$

therefore the probability required is

$$\frac{(m + n') (m + n' - 1) \dots \dots 1}{1 \cdot 2 \cdot 3 \dots m' \cdot 1 \cdot 2 \cdot 3 \dots n'} \\ \times \frac{(n + n') (n + n' - 1) \dots \dots 1}{(m + m' + 1) (m + m' + 2) \dots (m + m' + n + n' + 1)}$$

$$\begin{aligned}
& \times \frac{(m+1)(m+2) \dots (m+n+1)}{n \cdot (n-1) \dots 1} \\
& = \frac{(m'+n')(m'+n'-1) \dots 1}{1 \cdot 2 \cdot 3 \dots m' \cdot 1 \cdot 2 \cdot 3 \dots n'} \\
& \times \frac{(n+1)(n+2) \dots (n+n')(m+1)(m+2) \dots (m+m'+1)}{(m+n+2) \dots (m+n+3) \dots (m+n+4) \dots (m+m'+n+n'+1)} \\
& \text{If } (n+1)(n+2) \dots (n+n') \text{ be represented by } [n+1]^{n'} \\
& \quad (m+1)(m+2) \dots (m+m') \quad \quad \quad [m+1]^{m'} \\
& \text{and } (m+n+2)(m+n+3) \dots (m+m'+n+n'+1) \text{ by} \\
& [m+n+2]^{m'+n'}, \text{ this probability is expressed by} \\
& \frac{(m'+n')(m'+n'-1) \dots 1}{1 \cdot 2 \cdot 3 \dots m' \cdot 1 \cdot 2 \cdot 3 \dots n'} \frac{[n+1]^{n'} [m+1]^{m'}}{[m+n+2]^{m'+n'}}.
\end{aligned}$$

Which result in this form may be easily remembered, by observing that it is the same (with the difference of notation) as if the simple probability of drawing a white ball was $\frac{m+1}{m+n+2}$ and the probability of drawing a black ball was $\frac{n+1}{m+n+2}$.

Ex. 18. Let us suppose the sun to have risen 2000000 times, or days, then the probability that it will rise again is given by the preceding formula, by making $p = 2000000$, and $q = 0$, the probability required is $\frac{2000001}{2000001}$.

This probability, which is already very great, must be very considerably increased, if the discoveries of physical astronomy are taken into account.

49. If a white ball has been drawn p times, and a black ball q times, the probability of drawing a white ball in a future trial is $\frac{p+1}{p+q+2}$, and the probability of drawing a black ball is $\frac{q+1}{p+q+2}$; the greater p and q be-

come, the more nearly do these fractions coincide with $\frac{p}{p+q}$ and $\frac{q}{p+q}$, which are their limits when p and q are indefinitely increased. This theorem is the converse of Bernoulli's theorem, of which we gave a demonstration, p. 21.

Ex. 19. We said, that if a shilling was tossed into the air, the probability of its falling heads, or the reverse, twice running, was rather greater than $\frac{1}{2}$.

Let the probability of the shilling falling heads, be any of the following quantities:

$$\frac{1}{2} - ix, \quad \frac{1}{2} - (i-1)x, \quad \frac{1}{2} - x, \quad \frac{1}{2}, \quad \frac{1}{2} + x, \quad \frac{1}{2} + (i-1)x, \quad \frac{1}{2} + ix.$$

The number of hypotheses is $2i+1$, and the probability of the shilling falling heads twice running is

$$\frac{1}{2i+1} \left\{ \left(\frac{1}{2} - ix \right)^2 + \left\{ \frac{1}{2} - (i-1)x \right\}^2 \dots + \left\{ \frac{1}{2} - x \right\}^2 + \left(\frac{1}{2} \right)^2 \right. \\
\left. + \left(\frac{1}{2} + ix \right)^2 + \left\{ \frac{1}{2} + (i-1)x \right\}^2 \dots + \left\{ \frac{1}{2} + x \right\}^2 \right\}$$

$$= \left(\frac{1}{2}\right)^n + \frac{2x^n}{2i+1} \{1 + 2^i + 3^i \dots i^i\}$$

$$= \left(\frac{1}{2}\right)^n + \frac{2x^n}{2i+1} \left(\frac{i \cdot (i+1) \cdot (2i+1)}{2 \cdot 3} \right) = \left(\frac{1}{2}\right)^n + \frac{i \cdot (i+1) \cdot x^n}{3}$$

which probability is greater than $\left(\frac{1}{2}\right)^n$, the result when the shilling is supposed homogeneous.

Ex. 20. It follows, from what has preceded, that if an individual has made $m + n$ assertions, of which m are true and n are false; the probability of his telling the truth, in any case, is $\frac{m+1}{m+n+2}$, so far as we draw our conclusions from these assertions alone.

Let $\frac{m+1}{m+n+2} = v$, and let p be the *a priori* probability of an event

which he asserts to have taken place.

The event observed is the assertion by this individual that the event took place, of which the *a priori* probability is p .

If the event did take place, the individual tells the truth; the probability of the event on this hypothesis is pv .

The probability of the event on the contrary hypothesis is $(1-p)(1-v)$, therefore the probability of this hypothesis is

$$\frac{pv}{pv + (1-p)(1-v)}.$$

If $\frac{pv}{pv + (1-p)(1-v)} > p$, $v > \frac{1}{2}$.

Thus we see that when a witness asserts that an event has taken place, he renders the probability that it did take place greater than the simple probability of the event only when his veracity is greater than $\frac{1}{2}$, which result might have been foreseen.

Ex. 21. A witness asserts that out of a bag containing a thousand tickets, a given ticket, say No. 70, has been drawn, required the probability that this number was drawn.

Let the veracity of the witness be v , as before.

The event observed is the assertion by the witness that the given number was drawn, the probability of this event, on the hypothesis that the witness tells the truth, is $\frac{v}{1000}$, the probability of the event on the contrary hypothesis,

namely, that this ticket was not drawn, is $\frac{999}{1000}(1-v)$; but if the witness

is supposed to have no reason or inducement for choosing the No. 70 in preference to any other of the 999 undrawn numbers, this probability must be multiplied by $\frac{1}{999}$, which is the probability of the witness choosing this

number from the 999 undrawn numbers, so that the probability of the event on this hypothesis is $\frac{1-v}{1000}$.

The probability, therefore, of the first hypothesis is $\frac{v}{v + 1 - v}$, or v , the veracity of the witness.

Ex. 22. Two individuals, whose veracities are v and v' , assert that an event has taken place, of which the probability is p .

Two hypotheses are admissible, namely, that the event did take place, and that both the individuals tell the truth; or, that the event did not take place, and that both individuals lie; the probabilities of the assertions on these hypotheses are $v v' p$ and $(1 - v) (1 - v') (1 - p)$, therefore the probabilities of these hypotheses are

$$\frac{v v' p}{v v' p + (1 - v) (1 - v') (1 - p)} \text{ and } \frac{(1 - v) (1 - v') (1 - p)}{v v' p + (1 - v) (1 - v') (1 - p)}$$

respectively.

So if n individuals, whose veracities are $v_1, v_2, v_3, \dots, v_n$, assert the event to have taken place, the probability that it did take place is

$$\frac{v_1 v_2 v_3 \dots v_n p}{v_1 v_2 v_3 \dots v_n p + (1 - v_1) (1 - v_2) \dots (1 - v_n) (1 - p)}$$

If $n + 1$ individuals assert the event to have taken place, the probability that it did take place is

$$\frac{v_1 v_2 v_3 \dots v_{n+1} p}{v_1 v_2 v_3 \dots v_{n+1} p + (1 - v_1) (1 - v_2) \dots (1 - v_{n+1}) (1 - p)}$$

which is greater than the former probability if $v_{n+1} > \frac{1}{2}$, so that the assertion of the $n + 1^{\text{th}}$ individual increases the probability of the event arising from the testimony of the other n individuals, only when his veracity is greater than $\frac{1}{2}$.

Ex. 23. Two individuals, whose veracities are v and v' , assert that a given ticket has been drawn out of a bag containing a thousand tickets. The probability of the event on the hypothesis, that both the individuals tell the truth and that this ball was drawn, is $\frac{v v'}{1000}$; the *a priori* probability, that both the individuals lie and that the given ball was not drawn, is $(1 - v) (1 - v') \frac{999}{1000}$: if, however, these individuals have no inducement to choose the given ticket amongst the undrawn numbers, this probability must be multiplied by $\frac{1}{(999)^2}$, which is the probability of their both selecting the same number amongst the undrawn numbers; the probability, therefore, of the first hypothesis, namely, that the given ticket was drawn, is

$$\frac{v v'}{v v' + (1 - v) (1 - v') \frac{1}{999}}$$

If $v_1, v_2, v_3, \dots, v_n$ are the veracities of n individuals, who all assert that a given ticket was drawn, the probability that the given ticket was drawn is

$$\frac{v_1 v_2 v_3 \dots v_n}{v_1 v_2 v_3 \dots v_n + (1 - v_1) (1 - v_2) \dots (1 - v_n) \frac{1}{(999)^{n-1}}}$$

hence we see how prodigiously the probability of an event of this kind is increased by the concurrent testimony of many individuals. It must, however, be remarked, that the weight of the concurrent testimony of different individuals depends entirely upon the absence of inducement to lead these individuals to choose any one particular number, and upon the absence of collusion.

50. If we have no data to determine the veracity of an individual, and if he asserts an event to have taken place, of which the simple probability is p , in order to find the probability that the event took place, we must consider the probability of the event upon every hypothesis which can be formed; that is, we must suppose all values of his veracity between zero and unity to be equally probable, *a priori*. The probability, therefore, that the event did take place

$$= \int \frac{p v d v}{p v + (1 - v)(1 - p)} = \int \frac{p v d v}{1 - p + (2p - 1)v}$$

taken between the limits $v = 0$ and $v = 1$;

$$\begin{aligned} &= \frac{p}{2p-1} \int \frac{\{(2p-1)v + 1 - p - (1-p)\} dv}{1 - p + (2p-1)v} \\ &= \frac{p}{2p-1} \left\{ v - \frac{1-p}{2p-1} \log. (1 - p + (2p-1)v) \right\} + c \\ &= \frac{p}{2p-1} \left\{ 1 - \frac{1-p}{2p-1} \log. \left(\frac{p}{1-p} \right) \right\}. \end{aligned}$$

When $p = \frac{9}{10}$, this is equal to $\frac{9}{8} \left\{ 1 - \frac{1}{8} \log. 9 \right\} = .816363$ or about $\frac{9}{11}$.

Ex. 24. A jury consists of n individuals; let the probability of each separately giving a right decision be p , what is the probability that a unanimous decision is a correct one? Two hypotheses can be formed, namely, that the decision is a correct one, or the contrary; the event observed is a unanimous decision, and the *a priori* probability of this event on the first hypothesis is p^n , the *a priori* probability of the event on the second hypothesis is

$(1 - p)^n$, therefore the probability of the first hypothesis is $\frac{p^n}{p^n + (1 - p)^n}$,

which is greater than $\frac{1}{2}$, only when $p > \frac{1}{2}$. Therefore it is probable that a unanimous verdict is a correct one, only when it is probable that each jurymen considered separately will give a correct decision. The same rule holds *a fortiori*, when the verdict has been given by a majority only.

If $p = .9$ and $n = 12$, this probability is equal to $\frac{9^{12}}{9^{12} + 1}$.

A jury composed of $n - 2m$ individuals is correct.

Similarly, the probability that the decision of a majority is correct

$$= \frac{p^n + n p^{n-1} (1 - p) + n \cdot (n - 1) \cdot p^{n-2} (1 - p)^2 + \dots}{p^n + (1 - p)^n + n p \cdot (1 - p) \cdot \{ p^{n-2} + (1 - p)^{n-2} \} + \&c.}$$

The probability that a decision given by $n - 1$ is correct, is similarly

$$\frac{p^{n-1} (1 - p)}{p^{n-1} (1 - p) + p (1 - p)^{n-1}} = \frac{p^{n-2}}{p^{n-2} + (1 - p)^{n-2}}$$

and generally the probability that a decision given by $n - m$ of the jury is correct, is the same as the probability that a unanimous decision of

a jury composed of $n - 2m$ individuals is a correct one. If $n = 12$ and $p = .9$, the probability that a decision of a majority is a correct one by the preceding expression = $\frac{999438768179}{999508948516} = \frac{19519}{19520}$ nearly.

If p is unknown, the probability that a unanimous decision is a correct one must be found by taking the integral $\int \frac{p^n \cdot dp}{p^n + (1-p)^n}$ between the limits through which p may be supposed to vary, multiplied by $\int dp$ taken between the same limits.

51. The decision of the jury in this country can only be considered as that of a simple majority, and the probability that this decision is a correct one is small, unless the simple probability that each jurymen separately gives a correct one, is taken to be very great. If this probability is $\frac{2}{3}$, the probability of a correct decision is very little greater than $\frac{1}{3}$. The simple probability of any jurymen giving a correct decision cannot be supposed to be strictly the same for each jurymen composing the same jury, and it must also depend very much upon the nature of the question which is submitted to his determination. As this probability rests only on conjecture, we have considered the preceding questions relating to the decision of a jury with a view of showing how they might be solved if we were in possession of sufficient data rather than as laying any stress on the results obtained.

52. A bag contains a number of balls of i different colours:

m_1 of the 1st colour have been drawn and replaced,

m_2 " 2nd,

m_i " i^{th} ;

in $m_1 + m_2 + \dots + m_i$ trials; required the probability of drawing n_1 balls of the first colour, n_2 of the second, n_i of the i^{th} colour, in $n_1 + n_2 + \dots + n_i$ succeeding trials.

Let x_1 be the *a priori* probability of drawing a ball of the 1st colour,

x_2 " " " " 2nd,

x_i " " " " i^{th} ,

and let C be the coefficient of $x_1^{n_1} \times x_2^{n_2} \dots x_i^{n_i}$ in the development of $(x_1 + x_2 + \dots + x_i)^{n_1 + n_2 + \dots + n_i}$; then the probability of the observed event is $C \times x_1^{n_1} \times x_2^{n_2} \dots x_i^{n_i}$; the probability of the hypothesis that x_1 is the probability of drawing a ball of the 1st colour,

x_2 " " " 2nd,

x_i " " " i^{th} ,

is $C \times x_1^{n_1} \times x_2^{n_2} \times \dots \times x_i^{n_i}$ divided by the sum of all the values of which this fraction is susceptible; and if C_1 is the coefficient of $x_1^{n_1} \times x_2^{n_2} \dots x_i^{n_i}$, in the development of $(x_1 + x_2 + \dots + x_i)^{n_1 + n_2 + \dots + n_i}$, the probability of drawing n_1 balls of the 1st colour, n_2 balls of the 2nd colour, n_i balls of the i^{th} colour is the sum of all the values of which the quantity $C \times C_1 \times x_1^{(n_1 + n_1)} \times x_2^{(n_2 + n_2)} \dots x_i^{(n_i + n_i)}$ is susceptible, divided by the sum of all the values of which the quantity $C \times x_1^{n_1} \times x_2^{n_2} \dots x_i^{n_i}$ is susceptible.

If $x_1, x_2, \dots, x_i$ be supposed to vary from $x = 0$ to $x = 1$, and all these values are equally probable *a priori*, then the probability required is found by taking the integral

$$\int x_1^{(n_1 + n_1)} \times x_2^{(n_2 + n_2)} \dots x_i^{(n_i + n_i)} dx_1 dx_2 dx_i$$

between the limits $x_1 = 0, x_2 = 1 - x_1 - x_3 - \dots - x_{i-1}$

" " $x_{i-1} = 0, x_{i-1} = 1 - x_1 - x_2 - \dots - x_{i-2}$

" " $x_{i-2} = 0, x_{i-2} = 1 - x_1 - x_2 - \dots - x_{i-3}$

" " $x_1 = 0, x_i = 1$;

and dividing the quantity so obtained by the integral

$$\int^i x_1^{m_1} \times x_2^{m_2} \dots x_i^{m_i} \times dx_1 dx_2 \dots dx_i$$

taken between the same limits.

If $(m_1 + 1) (m_1 + 2) (m_1 + 3) \dots (m_1 + n_1)$ be represented by $[m_1 + 1]^{n_1}$

$(m_2 + 1) (m_2 + 2) (m_2 + 3) \dots (m_2 + n_2)$ " " $[m_2 + 1]^{n_2}$

$(\Sigma(m) + i) (\Sigma(m) + i + 1) \dots (\Sigma(m) + \Sigma(n) + i)$ by $[\Sigma(m) + i]^{i(n)}$

Σ being used as a sign of collection to denote that the sum is to be taken of all quantities which are represented by a general symbol, these integrations give for the probability required,

$$\frac{C [m_1 + 1]^{n_1} [m_2 + 1]^{n_2} \dots [m_i + 1]^{n_i}}{[m_1 + m_2 \dots \dots \dots + m_i + i]^{n_1 + n_2 \dots \dots \dots n_i}};$$

which is the same, with the difference of notation, as if the simple probability

of drawing a ball of the r^{th} colour was $\frac{m_r + 1}{m_1 + m_2 \dots + m_i + i}$. The probability

of drawing a ball of the r^{th} colour in one succeeding trial is

$$\frac{m_r + 1}{m_1 + m_2 \dots \dots + m_i + i}$$

53. One of the most interesting and useful applications of the theory of probabilities is the solution of questions connected with the duration of life and the calculation of the values of annuities and reversionary payments. The value of an annuity is the value of the sum of the annual payments made to an individual throughout his life.

Let $1 + \text{rate of interest} = \frac{1}{r}$, and let $p_{m,n}$ be the probability of a given individual aged m years living at least n years; the value of any sum s to be paid to him at the expiration of n years, neglecting discount, is the value of this sum multiplied by the probability of the individual being alive to receive it, which is equal to $s p_{m,n}$, this must be discounted in order to obtain its present value, which reduces it to $s r^n p_{m,n}$. If a_m is the value of an annuity of £1. to be received by an individual aged m years, the value of an annuity of £ s to the same individual is $s a_m = s \Sigma r^n p_{m,n}$.

54. When an individual insures his life at any office, the insurance company agrees to pay his executors a certain sum at his death, whenever that event may take place.

Let $q_{m,1}, q_{m,2}, q_{m,3}, \dots, q_{m,n}$ be the probabilities that an individual, aged m years, dies in the first, second, third, &c., or n^{th} year, then the value of £1. to be paid whenever he dies, discounted, is

$$r q_{m,1} + r^2 q_{m,2} + \dots \dots \dots + r^n q_{m,n};$$

but

$$q_{m,1} = 1 - p_{m,1}, \quad q_{m,2} = p_{m,1} - p_{m,2}$$

therefore the value of the insurance is

$$r(1 - p_{m,1}) + r^2(p_{m,1} - p_{m,2}) + \&c. \\ = r(1 + a_m) - a_m,$$

and the value of £ s to be paid when the individual dies is

$$s r(1 + a_m) - s a_m.$$

By means of this expression the value of an insurance at the age m may be deduced from the value of the annuity, and *vice versa*. A person insuring his life, instead of making one payment to the office, generally pays an

annuity which is called the *premium*, the value of this *premium* with present payment

$$= \frac{s r (1 + a_m) - s a_m}{1 + a_m} = r s - \frac{s a_m}{1 + a_m}$$

The value of any sum s to be paid if either of two individuals, aged m and m' years respectively, are alive after n years is

$$s r^n \{ 1 - (1 - p_{m,n}) (1 - p_{m',n}) \} \\ = s r^n p_{m,n} + s r^n p_{m',n} - s r^n p_{m,n} \times p_{m',n}$$

and the value of an annuity to be paid as long as either of two individuals, aged m and m' years, are alive

$$= s \{ \Sigma r^n p_{m,n} + \Sigma r^n p_{m',n} - \Sigma r^n p_{m,n} \times p_{m',n} \} \\ = s a_m + s a_{m'} - s a_{m,m'}$$

understanding by the symbol $a_{m,m'}$ the value of an annuity of £1. on the joint lives of two persons aged m and m' years.

If life is considered valuable in proportion to its duration the expectation which any individual has of life will be measured by the sum of the probabilities of his dying after each given age, that is by an annuity without interest.

Hence we have these expressions:

Expectation of life $= \Sigma p_{m,n}$

Value of annuity $s a_m = s \Sigma r^n p_{m,n}$

.....insurance of a given sum s in one payment

$$= s r (1 + a_m) - s a_m$$

.....premium of insurance of a given sum s

$$= r s - \frac{s a_m}{1 + a_m}$$

.....annuity during two joint lives

$$= s \Sigma p_{m,n} \times p_{m',n}$$

.....annuity to be paid as long as either of two individuals, aged m and m' years respectively, are alive

$$= s a_m + s a_{m'} - s a_{m,m'}$$

Thus if $m = 20$, $r = \frac{1}{1.03}$, $a_m = 20.1428$, according to Table III. for males.

Value of the single premium required for a male aged 20 to secure the payment of 1 at the end of the year in which the life shall fail

$$= \frac{21.1428}{1.03} - 20.1428 = .38419.$$

Value of the annual premium required for a male aged 20 to secure the payment of 1 at the end of the year in which the life shall fail

$$= \frac{1}{1.03} - \frac{20.1428}{21.1428} = .01817.$$

55. In calculating the values of annuities the labour is much diminished by observing that the probability of an individual aged m years living at

least n years is equal to the product of the probability of his living at least $n - q$ years, multiplied by the probability of an individual aged $m + n - q$ years living at least q years, or

$$p_{m,n} = p_{m,n-q} \times p_{m+n-q,q};$$

and therefore, putting

$$\begin{aligned} n-1 \text{ for } q, \quad p_{m+1,n-1} &= \frac{p_{m,n}}{p_{m,1}}, \\ a_{m+1} &= r p_{m+1} + r^2 p_{m+2} + \&c. \\ &= \frac{r^2 p_{m,2} + r^3 p_{m,3} + \&c.}{r p_{m,1}} \\ &= \frac{a_m - r p_{m,1}}{r p_{m,1}} = \frac{a_m}{r p_{m,1}} - 1. \end{aligned}$$

By means of this expression, which appears first to have been noticed by Mr. Barrett, the value of any annuity may be deduced from that which precedes or follows it.

Thus if $m=20$, $r = \frac{1}{1.03}$, $p_{m,1} = \frac{5707}{5765}$, according to Table II., and $a_m = 20.1428$, according to Table III.,

$$a_{21} = \frac{20.1428 \times 1.03 \times 5765}{5707} - 1 = 19.9580.$$

56. It has been seen that the values of annuities, reversionary payments, &c., consist of the sum of a number of separate payments. If these payments are calculated accurately at certain intervals the values of those which are intermediate may be interpolated by known methods.

In fact, if $y_0, y_1, y_2, \&c.$ are successive values of the variable y , and if $\Delta y = y_1 - y_0$, $\Delta^2 y = y_2 - y_1$, $\&c.$, and y_i be any value of y intermediate between y_0 and y_n ,

$$y_i = y_0 + i \Delta y + \frac{i(i-1)}{2} \Delta^2 y + \&c.$$

When the sum only of the values of y is required it is not necessary, however, to go through the labour of calculating each particular quantity in the series, it may easily be shown that this sum is equal to

$$\begin{aligned} &n(y_0 + y_1 + y_2 + \dots + y_n) \\ &+ \frac{n-1}{2} \{y_n - y_0\} - \frac{(n-1)(n+1)}{12n} \{\Delta y_n - \Delta y_0\} \&c. \end{aligned}$$

The problem appertains to what are called mechanical quadratures, and this method is similar to that made use of in summations which are required in calculating the perturbations of a comet. See the *Mécanique Céleste*, vol. iv. p. 206. In applications of this series to the calculation of annuities, reversionary payments, &c. $y_n, \Delta y_n, \&c. = 0$. The first term in the series of the values of y or y_0 is the value of a present payment $= 1$, if we neglect the term

$$\frac{(n-1)(n+1)}{12n} \{\Delta y_n - \Delta y_0\}$$

and the following, and suppose the values of the annual payments to be in arithmetical progression, the value of an annuity on the life of a person aged 20, to commence at the end of the first year, supposing $n=10$,

$$\begin{aligned}
 &= 10 \left\{ 1 + r^{10} p_{20,10} + r^{20} p_{20,20} + \&c. \right\} - \frac{9}{2} - 1, \\
 &= 10 \left\{ r^{10} p_{20,10} + r^{20} p_{20,20} + \&c. \right\} + \frac{9}{2}.
 \end{aligned}$$

57. In calculating annuities the values of the annual payments, except, perhaps, at birth, vary so gradually that the result thus obtained will be a sufficiently accurate approximation, and, probably, within the limits of the errors of which the values of p , that is of the table of mortality which is used, are susceptible; the correction, however, in all cases may be considered as constant for different tables of mortality, and may, therefore, be determined by calculating the annuity first accurately and afterwards by the approximate method from any table of mortality in which the deaths are given for every age, the difference between the two values so obtained will be the correction required.

The method of calculating annuities hitherto adopted by Dr. Price and other writers, has been, first to interpolate living between those which are actually given from ten years to ten years by the observations, to calculate probabilities of surviving each number of years from the numbers so interpolated, then to discount these probabilities so obtained, and finally to obtain the value of the annuity by adding together all these discounted probabilities. This labour, though diminished by means of the equation noticed by Mr. Barrett, is still unnecessary, and would lead to the same result as that given by the series of the last page. The same method is, we think, generally the simplest which can be applied to calculating annuities on two or more lives, and, in fact, to the summation of any series of which the law is too complicated to admit of the ordinary methods.

58. We have, as yet, said nothing with respect to the method of determining p and q , and this is a question of very considerable difficulty, whether as regards theory or practice.

Suppose 1000 infants to be carefully registered at birth, and the ages at which they die to be noted. If of these 900 are alive at the end of the first year the probability of an infant at birth living one year under similar circum-

stances would be nearly $\frac{900}{1000}$, or $\frac{9}{10}$; and if the number of Infants registered

were infinite, this would cease to be an approximation, the ratio of the number alive at the end of the first year to the whole number registered at birth would be exactly equal to the probability of an infant under the same circumstances living 1 year. The problem is, in fact, similar to the one we solved, page 33, when we supposed a bag to contain a number of balls of different colours, and that a certain number of drawings had been made. The different ages at which the individuals can die correspond to the different colours in the former problem.

If we suppose 101 ages at which deaths take place, that is, if we suppose n to vary from 0 to 100 in the values of $q_{m,n}$, $p_{m,n}$, and if $d_1, d_2, \&c. d_n$, are the number of the 1000 infants who have been observed to die in their first, second, and n^{th} years respectively, we have

$$\begin{aligned}
 q_{m,n} &= \frac{d_{m+n} + 1}{d_{m+1} + d_{m+2} + \dots + 101 - m} \\
 p_{m,n} &= \frac{d_{m+n} + d_{m+n+1} + \dots + 101 - m - n}{d_{m+1} + d_{m+2} + \dots + 101 - m}
 \end{aligned}$$

$$q_{n..} = \frac{d_n + 1}{d_1 + d_2 + \dots + 101}$$

$$p_{n..} = \frac{d_n + d_{n+1} + \dots + 101 - n}{d_1 + d_2 + \dots + 101}$$

in this case

$$d_1 + d_2 + \&c. \dots = 1000$$

$$q_{n..} = \frac{d_n + 1}{1101}, p_{n..} = \frac{d_n + d_{n+1} + \&c. \dots + 101 - n}{1101}.$$

59. Unfortunately, no registers of this kind have been kept, and we are obliged to have recourse to those sources which best supply this deficiency. If the population of any district were subject to no fluctuations arising from an influx of the inhabitants of the neighbouring countries, and if it were constant, that is, if the births and deaths were always the same and equal to each other, a register of the ages at which deaths took place would alone be wanted to determine the values of p and q for this place. For it is evident that if the *population were large* the probability of an individual dying at the n^{th} age would be equal to the deaths of persons at that age (n) divided by the births n years previously, but the births n years previously are, upon the hypothesis we have made, exactly equal to the present number of yearly deaths $= d_1 + d_2 \&c. = \Sigma d_n$, using the letter Σ as before,

and
$$q_{n..} = \frac{d_n}{\Sigma(d_n)};$$

and in the same way $q_{n..}$ may be determined.

The parish books, therefore, if they were accurately kept, and if the population were subject to the conditions we have mentioned, would furnish the information required, and they were used as a first approximation.

60. When the population is not stationary, the preceding results require modification.

Let d_n be the deaths observed at the age n in a given place, b_n the births in that place n years previously; then

$$q_{n..} = \frac{d_n + 1}{b_n + 101 - n}.$$

Let us suppose that the births m years ago were equal to the total number of deaths now, and that the births increase in a geometric progression, of which the common ratio is r ; then

$$b_n r^m = b_n = b_m r^m = \Sigma(d) r^m,$$

$$\therefore b_n = \Sigma(d) r^{m-n},$$

and
$$q_{n..} = \frac{d_n + 1}{\Sigma(d) r^{m-n} + 101 - n}.$$

m must be found, from the consideration that $\Sigma(q_{n..}) = 1$, which, in the present form of the equation, would be troublesome: the labour may be much simplified by observing, that when r does not differ much from unity, this value of $q_{n..}$ does not sensibly differ from

$$q_{n..} = \frac{d_n + 1}{(\Sigma(d) + 101 - n) r^{m-n}} = \frac{1}{r^m} \cdot \frac{(d_n + 1) r^n}{\Sigma(d) + 101 - n}$$

In this form $q_{n..} \cdot r^m$ may be calculated without any previous knowledge of m ; let the value of $q_{n..} \cdot r^m$ be called D_n ; then

$$r^n \sum (q_r) = \sum (D) = r^m$$

$$m = \frac{\log. \{ \sum (D) \}}{\log. r}.$$

In this manner, when r is taken at 1.005, m is found, from the Chester Observations, Table I., to be somewhere about 40. This result agrees, within the errors of observation, with the period given by an actual comparison of burials and baptisms.

61. The difficulty has been seen of applying mathematical reasoning to the valuation of risks which are connected with the duration of life, but the difficulty with fire and sea insurances is greater, from the number of circumstances which are necessary to be taken into account. If underwriters only insured against total losses, the question would be comparatively easy, for it would only be necessary to ascertain the number of houses out of a given number which are annually burnt to the ground, or the number of vessels which have been lost out of a given number making the same voyage. But by far the greater part of the claims on the underwriters arises from partial losses, as when a house is damaged by fire, or a cargo is partially injured by sea water, &c.

62. In insurances upon lives the observations which we have referred to furnish data which serve to ascertain the value of the risk, but in insurances against loss by fire and sea no similar data have been published, and so various and complicated are the contingencies to which they are subject, that it would be difficult to form any tables of the values of these risks. A register might be made out by each individual underwriter, or company of underwriters, showing the result of their respective experience, which would, doubtless, be useful to them in the conduct of their business. There are many important facts which might be collected and systematically arranged, the knowledge of which would also be useful to the underwriter. Thus a register of the weather in different parts of the world, for a sufficient number of years, would be some guide in ascertaining the relative value of the risk in maritime insurances, as far as it is affected by season. The Society for the Registry of Shipping appoints competent persons to survey every ship which enters any of the principal ports in this kingdom; their detailed report, which contains the name of the ship, captain, owners, her tonnage, the port where built, the materials of which she is made, &c., is published by subscription, and an office is kept for the purpose of posting it up to the latest period. Such a plan adopted in the principal maritime ports of foreign countries would, in the course of time, give a complete register of the commercial shipping of Europe. The preceding remarks apply to maritime insurances, and although the difficulty of obtaining sufficient data to determine the risk in fire insurances is considerable, it is not so great as in sea insurances.

It is foreign to our purpose to offer any further remarks upon these questions, the solution of which is the continual business of the underwriter, and which requires great skill and experience.

63. The principal use that has hitherto been made practically of the theory of probability has been in the solution of questions connected with the valuation of annuities and reversionary payments.

The *method of least squares*, which is of very extensive application in astronomy, was proposed by M. Legendre in 1805;* it has since been shown by Laplace† to be preferable to every other, when the number of

* *Nouvelles Méthodes pour la Détermination des Orbites des Comètes.*

† *Théorie des Probabilités.*

equations, which serve to determine any unknown quantities, exceed in number that of the unknown quantities themselves. Our limits do not permit us to give the analysis upon which this proof is founded, we shall, however, endeavour to explain shortly in what the method itself consists.

The data furnished by observation lead in general to equations of this form

$$a - bx + cy + \&c. = 0,$$

in which equation a , b , c , $\&c.$ are known quantities which vary from one equation to another. Each n^{th} observation gives an equation

$$a_n - b_n x + c_n y + \&c. = 0.$$

If the number of equations is equal to that of the unknown quantities x , y , z , they may be determined by linear elimination, but generally the number of equations exceeds that of the unknown equations, and the question arises, of which the solution is in some measure arbitrary, what system of equations, equal in number to that of the unknown quantities x , y , z , is most favourable for their determination.

The *method of least squares*, which Laplace has proved to be the most advantageous, consists in determining the quantities x , y , z , so that the

quantity

$$\begin{aligned} & \{ a_1 - b_1 x + c_1 y + \&c. \}^2 \\ & + \{ a_2 - b_2 x + c_2 y + \&c. \}^2 \\ & \quad \&c. \\ & + \{ a_n - b_n x + c_n y + \&c. \}^2 \end{aligned}$$

is a minimum.

For this purpose it is only necessary to differentiate this sum separately, with respect to each of the variables x , y , z , $\&c.$, and put the results separately $= 0$.

The equation, which is obtained by making x alone vary, is

$$\begin{aligned} & \{ a_1 - b_1 x + c_1 y + \&c. \} b_1 \\ & + \{ a_2 - b_2 x + c_2 y + \&c. \} b_2 \\ & \quad + \&c. \\ & + \{ a_n - b_n x + c_n y + \&c. \} b_n = 0, \end{aligned}$$

or

$$\sum a_n b_n' = x \sum b_n^2 + y \sum b_n c_n + \&c. = 0:$$

each of the quantities x , y , z , $\&c.$ furnishes a similar equation, and hence a system of equations results equal in number to those quantities, from whence they may be found by linear elimination. It is thus that many thousand observations may be made to concur in the determination of one element.

Suppose, for example, the question consists in determining with accuracy the elements of the orbit of a planet, after having obtained them nearly by a first approximation.

Let λ be the geocentric longitude observed, and suppose the uncertainty is with respect to two only of the elements e and ϖ , and let $\delta \lambda$ be the error of longitude, or the difference between the longitude of the planet observed at a given time, and that which is deduced by calculation from the approximate elements; δe , $\delta \varpi$ the errors of those elements; then by Taylor's theorem, neglecting the squares, $\&c.$ of δe and $\delta \varpi$

$$\delta \lambda = \left(\frac{d \lambda}{d e} \right) \delta e + \left(\frac{d \lambda}{d \varpi} \right) \delta \varpi$$

$\left(\frac{d\lambda}{d\epsilon}\right)$ and $\left(\frac{d\lambda}{d\varpi}\right)$, may be calculated directly, but it is better to infer them by assuming an arbitrary error $\delta\epsilon$, and by finding the corresponding error $\delta\lambda$.

If a and b be put for $\left(\frac{d\lambda}{d\epsilon}\right)$ and $\left(\frac{d\lambda}{d\varpi}\right)$ the n^{th} observation furnishes the equation

$$(\delta\lambda)_n = a_n \delta\epsilon + b_n \delta\varpi,$$

and the two equations which serve to determine $\delta\epsilon$ and $\delta\varpi$ by the *method of least squares* are

$$\Sigma (\delta\lambda)_n a_n = \delta\epsilon \Sigma (a_n)^2 + \delta\varpi \Sigma a_n b_n$$

$$\Sigma (\delta\lambda)_n b_n = \delta\epsilon \Sigma a_n b_n + \delta\varpi \Sigma (b_n)^2$$

If only one quantity has to be determined, this method evidently resolves itself into taking the mean of all the values given by observation.

64. We shall now, in conclusion, trace the theory of probability through the different stages of its progress, and mention the principal writers who have assisted in establishing its principles. The estimation of the probability of a future event, by enumeration of the cases supposed to be similarly circumstanced, does not appear to have been attempted until the early part of the seventeenth century; and the very elementary nature of the first problem of which the solution is on record, serves to show that the subject was then altogether new. It is contained in a fragment of uncertain date, written by the celebrated Galileo, who died in 1642. It was addressed to a friend who thought the fact that the points 9 and 10 can both be produced by six different combinations of numbers on three dice difficult to reconcile with the notorious preference given by gamblers to the latter number beyond the former. The difficulty is explained by Galileo, by taking into account the permutations of the component numbers, and the respective chances of these two numbers are thus shown to be as 25 to 27. A correct table is subjoined of the permutations of all numbers which can be thrown on three dice; and it is added, that the consideration of this table will serve for the solution of other problems of the same nature. All this must be admitted to belong to the infancy of the science, nor does it appear that Galileo thought the subject of sufficient interest to call for further inquiry.

65. The history of the theory of probability is generally made to begin several years later, when, in the year 1654, the two following problems were proposed by the Chevalier de Meré to Blaise Pascal.

1st. Two players want each a given number of points towards winning. If they separate without playing out the game, how should the stakes be divided between them?

2d. In how many trials is it an even wager to throw sixes upon two dice?

We are told in one of Pascal's letters to Fermat, that his answer to the latter question, that the odds are against twenty-four trials, and in favour of twenty-five, though undoubtedly correct, scandalized Mr. de Meré, "and made him declare loudly that the science of arithmetic is inconsistent with itself."* The Chevalier thought that the chances being in favour of throwing six in four trials with one die, on which are six different numbers, they ought also to be in favour of throwing two sixes in six times as many, or twenty-four trials with two dice, on which are six times as many, or thirty-six different numbers. Those who have read the preceding pages with any degree of attention, will readily perceive that the Chevalier (whose name it

* Opera Petri de Fermat. Tolosæ, 1679.

has become unusual to mention, without adding that he was a man of talent, but no mathematician) was thus comparing events which have no connection with each other. We shall have occasion presently to mention errors in the principles of this science committed under the sanction of a name of greater influence and authority among mathematicians.

66. The other problem (which afterwards obtained the name of the Problem of Points) appeared to Pascal of greater interest; he communicated it to Fermat, Roberval, and others; none of whom, but Fermat, returned him a satisfactory solution. The correspondence which passed on this subject between Fermat and Pascal appeared in 1679, in the posthumous edition of Fermat's works published at Toulouse, and is now also to be found in the complete edition of Pascal's works. Pascal began by considering the simplest case, in which one of two players, whom we will call A, wants one, and B, the other, two points of winning the game. He determined the required proportion from the consideration that if B win the next point, of which his chance is only $\frac{1}{2}$, these players would be in a condition of equality; and if they were then to separate, the stake ought to be equally divided between them; so that A's present share should be made up of half the stake corresponding to his equal chance of winning the next point, and one quarter corresponding to the present chance of his share, if B were to win the next; making $\frac{3}{4}$ in all. This mode of solution is very elegant, but there is some difficulty in applying the same principle to more complicated questions. It has been adopted by subsequent writers in examining the most difficult parts of the theory, but aided by a method of analysis very far superior to anything with which Pascal or his contemporaries were acquainted. In fact, Pascal was led into an error when he attempted to extend his method to the general problem, which occasioned a short controversy between him and Fermat, who had preferred the more laborious, but also more direct, method of enumerating all the possible ways in which the game might be terminated, and proportioning the division of the stakes according to the numbers which appear favourable to either party. Pascal found some difficulty in admitting that this method of Fermat's is good in every case, and confirmed himself in his mistake in consequence of an erroneous distribution which he made of the permutations of three letters, when he attempted to apply Fermat's method to the case of three players. Fermat pointed out the nature of his error, at least we may presume so from a letter of Pascal's, in which he retracts his former objection, saying, that Fermat's last remarks had been entirely satisfactory.

67. This correspondence was still unpublished, when Huyghens turned his thoughts to the theory of probability, and composed a short Latin treatise, "*De Ratiociniis in Ludo Aleæ*," first printed by Schooten in 1658 at the end of his "*Exercitationes Geometricæ*." This is the earliest regular treatise on the subject, which thenceforward continued to draw more and more the attention of mathematicians. Besides an examination in detail of Méré's questions, Huyghens' treatise contains the enunciation of the general and fundamental theorem of this branch of the science, that if any player have p chances of gaining a sum represented by a and q chances of gaining b , his *expectation* (a term then first introduced) will be rightly represented

by $\frac{pa + qb}{p + q}$. Elementary as this truth may now appear, it was not re-

ceived altogether without opposition.

68. In the year 1670, a Jesuit named Caramuel published the two first volumes of his course of mathematics, under the title of "*Mathesis Biceps*,"

in which he introduced a treatise on the theory of games at dice, which he called *Kubeia*, from the Greek word signifying a die. At the end of it he printed the whole of Huyghens's essay, professing to be ignorant whether it had been already published or not. Nicolas Bernoulli has characterised Caramuel's work as one continued blunder, and indeed this author has fallen into mistakes from which the reading of Huyghens's treatise ought to have preserved him. For instance, when proposing to determine the chances favourable to A and B; if the former (who is to begin) undertakes to throw six before the latter throws seven, upon two dice, his reasoning is as follows: A's chance of throwing six is $\frac{1}{6}$, and therefore, if the stake be 36, the value of his throw may be taken to be five, leaving thirty-one to B, whose chance of winning, if he have a throw, being equal to $\frac{1}{6}$, his first throw also may be bought off for $\frac{1}{6}$ of the remainder, or $5\frac{1}{6}$, leaving $25\frac{5}{6}$, still to be contended for. Caramuel's reasoning is so far correct; but instead of continuing to divide this remainder in the proportion of the chances of the two players, which would have led him to two infinite series, the sums of which would be the just proportions, he argued that the value of the first throw of each player being compensated to him by the share thus allotted to each out of the stake, this second remainder ought to be equally divided between them. Hence he deduced the shares of A and B, if they were to leave the game unplayed, to be respectively $17\frac{1}{3}$ and $18\frac{1}{3}$, instead of $17\frac{1}{6}$ and $18\frac{1}{6}$, which Huyghens had already deduced by a different and more correct analysis.

69. The *Journal des Sçavans* for 1679 mentions an essay published in the preceding November, by Sauveur, on the advantage of the banker at basset, a game of cards then much played in Paris, and celebrated for the duels it occasioned, to such an extent that it became necessary, solely on that account, formally to prohibit it from being played. This treatise was compiled at the request of the Marquis Dangeau, and brought Sauveur into great favour at court, where he was admitted to explain his theory to Louis XIV.*

70. It has been the misfortune of the science of probability, in consequence of the ready application made of its principles to games at cards and dice, that a prejudice has from the first existed against it as if ministering only to gambling and immorality, and available for no other purpose: accordingly the anonymous writer, who, in 1692, published the first English essay "Of the Laws of Chance," thought it necessary to protest in his preface that the design of his book was "not to teach the art of playing at dice, but to deal with them as with other epidemic distempers, and perhaps persuade a raw squire to keep his money in his pocket." This essay, which was edited, and is generally supposed to have been written, by Motte, the secretary of the Royal Society, contains a translation of Huyghens's treatise, and an application of his principles to the determination of the advantage of the banker at pharaon, hazard, and other games, and to some questions relating to lotteries. The body of the work does not contain any new principle, but there are some remarks in the preface, which, considering the time at which they were written, deserve attention, and show how justly the author had apprehended the nature of his subject. "It is impossible," says he, "for a die with such determined force and direction not to fall on such a determined side, only I do not know the force and direction which make it fall on such a determined side, and therefore I call that chance, which is nothing but want of art."—"There are very few things which we know, which are

* *Histoire de l'Académie*, 1716.

not capable of being reduced to a mathematical reasoning; and when they cannot, it is a sign our knowledge of them is very small and confused; and where a mathematical reasoning can be had, it is as great folly to make use of any other, as to grope for a thing in the dark when you have a candle standing by you."—"There is likewise a calculation of the quantity of probability founded on experience, to be made use of in wagers about any thing. The yearly bills of mortality are observed to bear such proportion to the live people as 1 to 30 or 26; therefore it is an even wager that one of thirteen dies within a year (which may be a good reason, though not the true, of that foolish piece of superstition,) because at this rate if 1 out of 26 dies, you are no loser."

71. Long before mathematics had been applied to this science, Kepler had formed the same accurate notion of the real meaning of chance as is here expressed by Motte. In his dissertation on the new star which appeared in 1604, after mentioning that some were of opinion it came by chance, and illustrated their meaning by supposing a set of dice to be thrown an infinite number of times, in which it would necessarily happen (according to them) that any required number would at last be thrown, he says that, even in the case adduced, those are very unthinking who look upon the events as entirely without a cause. "Why does six fall in one throw and ace in another? Because this last time the player took up the die by a different side, shut his hand upon it differently, shook it, threw it in a different manner, or because the wind was blowing differently upon it, or it fell on a different part of the board. There is nothing in all this, which is without its proper cause, if any one could investigate such niceties."*

72. The bills of mortality, mentioned in Motte's book, are registers which began to be kept in 1592, of the annual number of deaths in the city of London, which, with some intermission between 1594 and 1603, have been regularly returned to the present time. They were first intended to make known the progress of the plague; and it was not till 1662 that Captain Graunt, a most acute and intelligent man, conceived the idea of rendering them subservient to the ulterior objects of determining the population and growth of the metropolis; as before his time, to use his own words, "most of them who constantly took in the weekly bills of mortality, made little or no other use of them than so as they might take the same as a text to talk upon in the next company; and withal, in the plague time, how the sickness increased or decreased, that so the rich might guess of the necessity of their removal, and tradesmen might conjecture what doings they were like to have in their respective dealings." Graunt was careful to publish with his deductions the actual returns from which they were obtained, comparing himself, when so doing, to "a silly schoolboy, coming to say his lesson to the world (that preevish and tetchie master,) who brings a bundle of rods, wherewith to be whipped for every mistake he has committed." Many subsequent writers have betrayed more fear of the punishment they might be liable to on making similar disclosures, and have kept entirely out of sight the sources of their conclusions. The immunity they have thus purchased from contradiction could not be obtained but at the expense of confidence in their results.

73. These researches procured for Graunt the honour of being chosen a fellow of the Royal Society, and, to pass over Sir Wm. Petty's "Observations," as bearing less directly on our subject, were undoubtedly the cause which led Halley to consider the duration of human life, as he himself owns, in the

* *De Stellâ novâ. Præge, 1606.*

paper published in the Philosophical Transactions in 1693. In this celebrated paper, from which we must date the commencement of real knowledge on the subject of life annuities and insurances in this country, Dr. Halley has made choice of a register of deaths which had been kept at Breslau, in Silesia, and which had been then recently communicated by Neumann (probably at Halley's request,) through Justell, to the Royal Society, in whose archives it is supposed that copies of the original registers are still preserved. Before continuing our notice of this very interesting branch of the subject, we shall mention some other important works which appeared about the same time.

74. James Bernoulli had shown that he was not inattentive to the progress of this science by a problem which, according to the fashion of that time, he had published in the *Journal des Sçavans* for 1685, in the form of a challenge to his contemporaries. This problem was to determine the chances of A and B, who are each to score a certain number of points thrown on the dice, A beginning with one throw, B following also with one; A then being allowed two, and B three, and A three, and B three, and so on till the conclusion of the game. Leibnitz answered the question, and undertook to divine the analysis which had conducted Bernoulli to the solution given by him, without demonstration, in the *Journal de Leipsic* for May, 1690.

75. There is also a treatise by Leibnitz, on Complexions, or, as we now more commonly call them, Combinations, but which was not written with any reference to the science of chances, in which this theory is so pre-eminently useful.

The distinctive names which Leibnitz adopts, of combinations, conternations, conquaternations, &c., to express what we now call combinations two by two, three by three, &c., are no longer in use, except among German writers, where we still meet with the terms binions, ternions, quaternions, &c. Leibnitz mentions Clavius as having been the first who gave, in 1583, a clear view of this theory, "not being able to find any traces of it in the Arithmetic of Cardan, to whom Schwenter refers it." Schwenter probably alluded to Cardan's book, "*De Proportionibus*," in which the figurate numbers are mentioned, and their use shown in the extraction of roots, as employed by Stifel, a German algebraist, who wrote in the early part of the sixteenth century.

76. It is not necessary to do more than mention an essay, by Craig, on the probability of testimony, which appeared in 1699, under the title of "*Theologię Christianę Principia Mathematica*." This attempt to introduce mathematical language and reasoning into moral subjects can scarcely be read with seriousness; it has the appearance of an insane parody of Newton's *Principia*, which then engrossed the attention of the mathematical world. The author begins by stating that he considers the mind as a movable, and arguments as so many moving forces, by which a certain velocity of suspicion is produced, &c. He proves gravely, that suspicions of any history, transmitted through the given time (*cæteris paribus*), vary in the duplicate ratio of the times taken from the beginning of the history, with much more of the same kind with respect to the estimation of equable pleasure, uniformly accelerated pleasure, pleasure varying as any power of the time, &c. &c.

77. An anonymous essay in the Philosophical Transactions of the same year, and of not much greater value, may perhaps be attributed to the same author. The theory there laid down is, that a fraction of the doubt which always remains as to the truth of a narrated fact, after any number of concurrent witnesses, is always removed by an additional testimony. This obviously false theory was taken up at a much later period, by Biquilley, in

a work entitled "*Du Calcul des Probabilités*," and by Condorcet in the article "*Probabilité*," in the French Encyclopædia.

78. James Bernoulli was employed in preparing a copious work on the science of chances, till his death in 1705, by which its appearance was delayed during ten years, after which his nephew, Nicolas Bernoulli, found leisure to superintend its publication, though in an unfinished state. In the mean time Montmort published his celebrated work, "*Essai d'Analyse sur les Jeux de Hazard*," the most extensive of the sort which had till then appeared, in which the conditions of all the principal games then in vogue are discussed at considerable length, and the theory of combinations extended and enriched with several new theorems. Immediately after the publication of Montmort's book, Demolivre, a Frenchman naturalized in England after the revocation of the Edict of Nantes, inserted, in the Philosophical Transactions for 1711, a short essay entitled "*De Mensura Sortis*," which, in 1716, he published in a greatly enlarged form, under the title of "*The Doctrine of Chances*." This work is far superior, both in research and elegance, to all which had preceded it; the collection of problems which it contains is far too extensive to admit any complete notice of it being given in this place: it will be sufficient to instance the doctrine of recurring series, and the theorems on the duration of play, which are to be met with in it for the first time, to show how much farther Demolivre carried his inquiries than those who had written on the subject before him. Montmort, who had been a personal friend of Demolivre, thought he had some reason to complain of the manner in which Demolivre spoke of his methods, and a coolness existed in consequence for some time between them.

79. The treatise by James Bernoulli, mentioned above, which was published in 1715, by his nephew Nicolas, and entitled "*Ars Conjectandi*," may be considered as belonging to the earlier period, at which unquestionably it was written. It is divided into four parts; the first consisting of Huyghens's treatise, with a commentary on most of the propositions. The second contains the abstract theory of combinations, in which are many new and elegant results; amongst others, the expression for the sum of the a^{th} powers of the natural numbers, in which series occur for the first time those remarkable coefficients since become so famous, under the name of Bernoulli's Numbers. A less profitable labour, which is also to be met with in this part of the work, is the curious analysis of the permutations of the celebrated Latin verse, *Tot tibi sunt dotæ, Virgo, quot sidera cælo*, which are determined to be 3312 in number, without transgressing the laws of Latin metre. It does not appear that James Bernoulli intended to publish this analysis, which was found by his nephew among his loose papers. The third part gives the application of the preceding principles to a variety of questions. The following problem deserves notice, because Bernoulli has given a false, though plausible solution of it, together with the true one, in order, as he says, to show what care is necessary to avoid error in the discussion of these questions. A is to throw a die, and to repeat his throw as many times as the number thrown the first time. If the sum of the points given by the latter set of throws be less than 12, A loses; if more, he wins; if they equal 12, he takes half the stake. His expectation is required. The true value of his

expectation is found to be $= \frac{15295}{31104}$: rather less than $\frac{1}{2}$, the false solution is

as follows:—A has $\frac{1}{2}$ chance of throwing an ace at the first trial, in which case he will have but one throw to reckon upon, and as this may equally give him any number from 1 to 6, his chance from it may be reckoned at

$$\frac{1 + 2 + 3 + 4 + 5 + 6}{6} = 3\frac{1}{2}. \text{ In the same manner, if he throw a deuce}$$

the first time, he cannot, in the two throws which this secures to him, score less than two or more than 12, and it is easy to see that the chances of each number equidistant from the mean throw, 7, are equal; 7 is, therefore, his expectation on this supposition. If 3, 4, 5, or 6 be thrown the first time, his expectation on each will similarly be found to be the means between 3 and 18, 4 and 24, 5 and 30, 6 and 36, or $10\frac{1}{2}$, 14, $17\frac{1}{2}$, and 21 respectively, so that the expectation of his throws will be $\frac{1}{4} \{ 3\frac{1}{2} + 7 + 10\frac{1}{2} + 14 + 17\frac{1}{2} + 21 \} = 12\frac{1}{2}$ points, from which it would appear that the odds are in his favour. From the manner in which Bernoulli dwells upon the plausibility of this solution, it seems not improbable that he had been himself deceived in the first instance, by the erroneous view which he here exposes to his readers. The error consists in not multiplying each chance separately by the gain or loss it could occasion to the player. The fourth part of Bernoulli's book, which had been expected with the greatest impatience, but which unfortunately was left incomplete at the time of the author's death, was intended to contain an application of the theory of probabilities to the examination of questions connected with civil and domestic life. Imperfect as this has been left, it must, undoubtedly, be considered as the foundation of whatever has been since done in this branch of the science. Bernoulli seems to have been the first to introduce the term "moral certainty," on which we have already remarked in the body of the treatise. He also, in imitation of Aristotle,* distinguishes between what he calls free and casual contingencies, classing under the former all those contingent events which depend on the will of a rational being. He also inculcates strongly the fundamental principle, from the neglect of which so much error and confusion have arisen, that "contingency or chance has reference merely to the state of our knowledge." After explaining the principal rules by which we should be guided in our investigations, he proceeds, among other things, to examine the method of determining probabilities *a posteriori*, that is to say, by often-repeated experiment: and shows, in the noted theorem which still bears his name, that the probability of attaining to the knowledge of the probability of an unknown event constantly increases with the number of experiments made upon it, so that we can always, by multiplying our experiments, reach a degree of probability as near certainty as we choose to fix upon, that the error of our estimation lies within given limits. With this proposition the work abruptly terminates. At the end of Bernoulli's book is an anonymous letter on the game of tennis, the author of which is not certainly known, but the theory, language, and notation, strongly mark it to belong, if not to James Bernoulli himself, at any rate to some one trained in the same school, and fully imbued with his ideas and opinions.

80. The first great step beyond what Bernoulli had suggested, showing the use to be made of experiments in estimating unknown probabilities, is contained in a posthumous paper by Bayes, inserted in the *Philosophical Transactions* for 1763, through the means of Dr. Price, so well known by his own publication on the subject of annuities. The problem which Bayes proposes to solve in this paper is the following:—Given the number of times in which an unknown event has happened and failed. Required the chance that the probability of its happening in a single trial lies somewhere between any two degrees of probability that can be named. When disencumbered of the geometrical form under which it was then the fashion to represent inte-

* *Physics*, lib. ii. cap. 6.

grals, Bayes's theorem is in substance, that if α and A be any two fractions between 0 and 1, the probability that the happening of an event which depends on unknown causes, but which has been already observed to happen p times exactly in $p + q$ experiments, has a degree of probability not greater than α ,

and not less than A , will be equal to $\frac{\int \alpha^p \cdot (1-x)^q \cdot dx}{\int A^p \cdot (1-x)^q \cdot dx}$, the integral in the

numerator being taken between the limits α and A , and in the denomination between 0 and 1. This theorem rests on the more elementary one, that the probability of the existence of a supposed cause of any observed event is proportional to the probability of the event, derived from the supposition of that cause being known to be true. The rest of the paper is taken up with different methods of approximating to the values of this integral within required limits, with which we need not here occupy ourselves. Bayes, or perhaps we should rather say Price, seems to have confounded the probability thus determined, with the probability that an event which has been already observed m times in $p + q$ experiments, will happen again. The difference between the two is obvious; and the reader has already seen the process for determining the latter.

81. The celebrated question, known as the Petersburg Problem, has been already mentioned: this name was given, on account of its having been proposed by Daniel Bernoulli in the Petersburg Transactions; much of the discussion it occasioned might have been spared if the real meaning of the results of the calculations of probability had been kept steadily in view. The difficulty of that question was supposed to consist in this, that no person could be supposed willing to pay the amount which the condition of the game pointed out as equal to his expectation, which after all amounts to no more than saying, that a game can be contrived of too ruinous a nature for the taste even of the most inveterate gamester. It has been well remarked by Bullon, that the science of probabilities never professed to make the condition of a gambler the same as if he did not play; it only indicates the events of which we have most reason to expect the recurrence. Condorcet took away everything appearing paradoxical from the result, by an observation he made in a memoir on this subject in 1784. "It may often happen," says he, "that a reasonable man A will refuse to give B a sum b for the chance

n of gaining α , although α be greater than $\frac{b}{n}$; and the reason may be, because

A has not the opportunity of repeating the venture often enough to repair the loss which may accrue to him in a single trial, and because the sum ventured may be so great that its loss would occasion him an inconvenience, not at all counterbalanced by the advantages he could derive from his contingent gain.* These are motives for inducing A to refrain from venturing, but cannot be made elements of the calculation as between him and a speculator B on the opposite event. No underwriter diminishes, or ought to diminish, his premium, on account of the small fortune of the party whose indemnity he guarantees.

82. There have been, however, some writers of great celebrity, who have taken an opposite view of this question; and although there can be no doubt of the fallacy of their reasonings, a notice of them must not be omitted in an historical account. D'Alembert instanced this Petersburg problem as tending to throw doubt on the universally admitted rule, that in every game the

* Histoire de l'Académie Royale, 1784

deposit ought to be inversely proportional to the risk, which therefore he proposed to examine. The result of his examination was, that, in his opinion, a very small probability should be considered as none, and might be entirely disregarded. He illustrates this, by supposing "Peter to play with James, on this condition, that if a tossed halfpenny fall head, in the hundredth toss, and not before, he is to receive from James 2^{100} crowns, in which case the ordinary rule would determine Peter to give James one crown at the beginning of the game. I say, Peter ought not to give this crown, because he will lose it *certainly*,—because head will fall *certainly* before the 100th toss, although *not necessarily*."* Again, he says, "We must distinguish between what is *metaphysically*, and what is *physically* possible. In the first class is everything whose existence does not imply an absurdity; in the second, everything whose existence not only does not imply an absurdity, but even implies nothing too extraordinary, and beyond the daily course of events. It is *metaphysically* possible that two sixes may be thrown on two dice a hundred times in succession; but it is impossible *physically*, because it has never happened, and never will happen." In the same memoir he advances the opinion, that the oftener an event has already happened in succession, as, for instance, the oftener a halfpenny has already successively fallen head, the less is the probability that it will fall head in the next trial. It is rather singular that he did not from the first observe, that the extension of this principle to its utmost limit, namely, to the case in which the halfpenny should always have fallen head, would oblige us, according to his own rule, to class the arrival of tail among the things physically impossible, which never have happened, and which therefore we have no reason to believe ever will happen; and yet, according to his present argument, it will be precisely in this case that tail will be most likely to happen in the next trial. A sounder principle might have suggested to him, that so far as our judgment is determined solely by the supposed repetition, we should be disposed, on that very account, to expect rather the recurrence of head, the oftener it has already appeared, because that very inequality would seem to point out an inequality in the sides favouring that event. The real cause why this effect in ordinary cases is not produced, is that we tacitly refer to, and are influenced by, the great number of times in which head and tail have followed each other indiscriminately, as well as to all the other reasons we have for believing the two sides similarly circumstanced, and the probability arising from this of the perfect indifference of the sides is far from being outweighed by the results of a few sequences. Another error, not less extraordinary, was made by the same celebrated writer in the consideration of repeated experiments. If a player undertook with a halfpenny to throw head in two trials, D'Alembert observed that there were but three possible cases: head in the first trial; tail in the first, and head in the second; tail in both. He, therefore, asserted that the chance in favour of the player should be taken at $\frac{3}{4}$, and not $\frac{3}{8}$, according to the ordinary rule, in which the combination of head thrown twice is taken into the account, "because as soon as head is thrown the first time, it is as useless as ridiculous to throw the piece again; for the result of the second throw has no effect upon the game, and is as foreign to it as if, instead of throwing the piece again, the players had gone to Rome." D'Alembert's mistake lies in supposing that his opponents were not as much aware of this as himself. It is true that in this game there would be only three possible cases, but they would not be similarly circumstanced, which it is necessary that they should be, before an enumeration of them can furnish us

with a measure of probability; and the superior probability of head falling the first time, as compared with either of the other two combinations mentioned by him, can be accurately allowed for only by taking the sum of all the ways in which one of two conflicting events may occur in two experiments. As to D'Alembert's farther observations on the possible difference in the law of facility of sequence of any set of events, the only answer that can be given is, that when any such difference is observed, it ought undoubtedly to form an element of the calculation of probabilities; but to suppose, as he seems to have done, that until such law is determined we are unable to estimate them, is to misapprehend entirely the meaning of the results we profess to deduce from it.

83. Laplace inserted several memoirs on the subject of probabilities in the Memoirs of the French Academy, which he afterwards embodied in his splendid work, "*Théorie Analytique des Probabilités*," in which he also gave the calculus of generating functions. The principal application which he there makes of it, is to the solution of the equations of differences to which he reduces the questions of probability, by considering how the chances vary at each succeeding step. This is, in fact, the method, of which one of the simplest instances may be seen in Pascal's solution of the problem of points, although it is there put in rather a different form. Laplace's work contains the application of the theory to a variety of most intricate and interesting questions; and independently of the results he has obtained, this book is in the last degree valuable, from the specimens of refined and beautiful analysis it affords. Besides the authors here mentioned, a great number have composed works on the subject of probability, not very remarkable for the introduction of new principles or methods. The principal attention of English writers has been directed to compilations on the subject of annuities; and it is very much to be regretted that, with a few exceptions, a wanton and barbarous scheme of notation should conceal whatever may be valuable in their writings, nearly as much as if they were written in an unknown language.

84. The first complete tables of life annuities constructed in this country from Halley's and Demoiivre's theory were by John Richards, of which the edition in the British Museum is dated 1730.* It contains the following short but curious historical sketch of the erroneous methods it was intended to supersede. "In valuing three lives absolute in copyhold estates, the general rule was, formerly, to reckon it as a lease of twenty-one years certain, which, by the tables for that purpose, at 5 per cent., is worth, in ready money, 12·82 years value and no more for three lives, the first of which they esteemed worth 6 years, the second 4, and the third 2·82; so that to renew two lives in reversion of one would cost 7 years, or one in reversion of two, three years' value. And this was the constant expectation, what age soever the life or lives *in esse* were of at the time of renewing. Whether this estimation of the value of leases arose from the Act of 32 Henry VIII., or was in use before, I know not; but it is there enacted, that a lease for more than twenty-one years, or three lives, is void: by which it seems as though three lives and twenty-one years were reckoned an equal duration,† the contrary of which was very evident even before any experiments were made concerning the duration of life, and therefore this way of computing was corrected by another, which is likewise in several respects erroneous. By this other method (which is still in practice) a lease for one life may be reckoned equivalent to one of 9, 10, 11, or 12

* Gentleman's Steward Instructed. London, 1730.

† It seems more likely that the framers of the Act were guided by the pre-existing popular prejudice, than that they gave rise to it.

years, &c.; that for two lives at 17, 19, 21, or 23 years, &c.; that for three lives as a lease of 24, 27, 30, or 33 years, &c.; and though this latter method is a little more plausible than the former, on account of the steward's liberty of choosing which of these proportions he pleases, yet I cannot see any analogy that this bears to the reason of the thing. So that at best it is but only groping in the dark."

85. The old tables, to which Richards here alludes, are frequently referred to in the treatises of that day, under the name of *Æroid's Tables*: the time at which their author lived was not even then accurately known, but is conjectured to have been about the time of Henry VIII.'s reign, as the interest of money when they were compiled was rather more than 10 per cent. The slightly improved method which Richards mentions as then still in practice, was suggested in an anonymous treatise entitled "*Tables for renewing and purchasing Leases of Lives*," first published at Cambridge about 1685. This is the book which is often cited as Newton's "*Treatise on Life Annuities*," but with which he was no otherwise concerned than as it bears his approbation as Lucasian Professor on the title-page, which is couched in the following terms:—"The method of this book is correct, and the numbers computed with sufficient accuracy, as I judge from re-calculating several of them."* There are no other traces that Newton meddled with this subject. In this treatise *Æroid's tables* are said to have been calculated at 11*l*. 3*s*. 6*d*. per cent. The tables which Richards published are calculated at different rates, from 4 to 8 per cent., and are given for all ages, from five to five years on one life, and from ten to ten on two, and on three joint lives. Since that period the principal improvement of such tables has consisted in more careful and extensive registers of deaths to furnish the requisite data for their construction: the only addition to their theory has been in the suggestion, that such registers ought only to be considered as furnishing the results of a number of experiments: consequently, that the ratios given by them ought not to be immediately employed as probabilities *à priori*, but used as in the theory given in page 37.† An excellent account may be found of the authors who have treated of questions connected with annuities in the article *MORTALITY*, in the supplement to the "*Encyclopædia Britannica*."

86. The first tables of mortality were, in fact, formed by Dr. Halley from the registers of Breslau in Silesia, and are given in the *Transactions of the Royal Society* for 1693, art. 1. The next author who treated of this subject is William Kerseboom, who published at the Hague, in 1730, a tract, entitled, "*Eerste Verhandelng tot een Proeve om te weeten de probable menigte des volks in de provincie van Hollandt en Westvrieslandt*." An account of this work is given by Mr. Eames in the *Philosophical Transactions* for 1738. In 1742 Kerseboom published two other tracts upon the same subject, an account of which is given by Mr. Van Rixtel in the *Philosophical Transactions* for 1743. Kerseboom's table of mortality was formed from registers of many thousand life annuitants in Holland and West Friesland, which had been kept there from 125 to 130 years previous to the date of his publication.

87. M. Desparcieux published, in 1746, his "*Essai sur les Probabilités de la Durée de la Vie Humaine*," in which he gave several tables of mortality, constructed from the lists of nominees in the French tontines and from the

* *Methodus hujus libri recte se habet: utmerique, ut ex quibusdam ad calculum revocatis, judicio, satis exacte computantur.* Is, Newton, Math. Prof. Luc.

† When insurances on lives began to be established in the beginning of the last century, their true principles were so little understood or acted upon, that every insurer, of whatever age he might be, paid the same premium, the only restriction being, that his life should lie between five and sixty years.

mortuary registers of different religious houses. The Northampton Tables, as they are called, were long the only tables in use in this country; they were given by Dr. Price in his "Observations on Reversionary Payments," which was published in 1771: the following extract will serve to explain the manner in which they were formed, vol. 2, p. 94. "In this town (Northampton), containing four parishes, namely, All Saints', St. Sepulchre's, St. Giles's, and St. Peter's, an account has been kept ever since the year 1741 of the number of males and females that have been christened and buried, dissenters included, in the whole town. And in the parish of All Saints, containing the greatest part of the town, an account has been kept since 1735 of the ages at which all have died there.

Christened	{ males 2152 females 2066 }	4220	Buried	{ males 2377 females 2312 }	4689
Of these died					
Under two years of age	1529		50 and 60	...	384
Between 2 and 5	362		60	...	70...378
5	10	261	70	...	80...358
10	20	189	80	...	90...199
20	30	373	90	...	100... 22
30	40	329			
40	50	365	Total	...	4689.

"The XVIIth Table in this volume is the genuine table of observations for Northampton, from which may be calculated the true probabilities and values of lives in that town." To the preceding paragraph is added this note. "In the fourth edition of this treatise the following corrections were made in this table; first, the table printed in the first three editions having been formed from the Northampton bills for 36 years, this table was rendered a little more correct in consequence of being formed from the same bill for 46 years. Secondly, the bills give the number dying annually between 20 and 30 greater than between 30 and 40; but this being a circumstance which does not exist in any other register of mortality, and, undoubtedly, owing to some accidental and local causes, the decrements were made equal between 22 and 40; preserving, however, the total of deaths between 20 and 40 the same that the bills give them. Thirdly, the bills giving only the totals of deaths under 2 years of age and between 2 and 5, the proportions of deaths for every particular year between 2 and 5, and for every quarter of a year after birth till one year of age, were made the same nearly that the Chester register makes them."

Such are the alterations which Dr. Price made in the data which were presented to him, and he reduced the table of mortality to the radix 11650; why he chose this number in preference to any other we have not been able to discover. Dr. Price has also neglected to inform us what method he made use of to interpolate the living at those ages, between every ten years, which are not given by the observations.

68. Such is the history of the Northampton Tables. We shall now quote Dr. Price's observations on the Chester Tables.

"Chester is a healthy town, of moderate size, where the births had for many years a little exceeded the burials; and the register to which I refer had the particular advantage of being under the direction of Dr. Haygarth, its founder as well as conductor. As it gives an accurate account of the distempers of which all the inhabitants die in every season and at every age, it contains much physical instruction; but my views lead me only to take notice of that part of it which gives the law according to which human life

"wastes in all its different stages, both among males and females. Concerning these tables it is necessary I should make the following observations.

"The table for females must be considered as particularly correct, because the number of females born and buried in Chester are very nearly equal. On the contrary, the number of males born being about an eighth greater than the number buried, it follows that, in the table of decrements for males the number of the living, and, consequently, the probabilities of living at every age for the last 10 or 15 of the first years of life, must be given too low."

89. Dr. Price characterises Dr. Haygarth as an able and ingenious physician, and it appears that he made a survey of the ten parishes of Chester with great care in 1774, at which time the population consisted of 6697 males and 8016 females.

The table of mortality formed by Dr. Haygarth, from the observations at Chester, is given at length in Dr. Price's *Treatise on Reversionary Payments*, vol. ii. It distinguishes the sexes, which the Northampton Table does not; it contains 4006 observations, while Carlisle only furnished to Mr. A. Milne 1840, a number too small to admit of subdivision. The Chester observations were, probably, communicated to Dr. Price after those at Northampton, which may have been the reason why he made more use of the latter, probably also as they were rather the more favourable of the two, he wished to keep on the safe side. Dr. Haygarth has given different tables of observations in the *Transactions of the Royal Society*, in which the deaths are classed from 5 years to 5 years, and the diseases by which they were occasioned are also stated. As Dr. Haygarth was a physician practising in Chester at the time he collected these observations, he had great opportunities of obtaining exact information. Dr. Haygarth states distinctly, that all the numbers dying at every age were taken just as the register gave them, except in the case of 22 females above the age of 80, of whom the age was not closely specified.

It is much to be regretted that no registry of births, marriages, and deaths exists in this country, which would furnish very valuable statistical information. The act of the 52 Geo. III., for the better regulating and preserving parish and other registers of births, baptisms, marriages, and burials in England, has, indeed, a clause by which any person making false entries, or false copies of entries, is to be adjudged guilty of felony, and transported for fourteen years. Another clause which follows immediately after, directs that one half of the penalties levied in pursuance of this act shall go to the informer and the remainder to the poor of the parish. The returns of the weekly burials, which are made by the clerks within the bills of mortality, do not appear to be sent regularly to the parish clerk's office, so that it is difficult to ascertain the effect of the seasons or the weather in producing deaths. In these returns the number of marriages is not stated, nor are the burials of males and females discriminated.

90. We have before alluded to the error which arises in a table of mortality considered as furnishing the probabilities of life, from the supposition that the population has been stationary during the century previous to the observation, and we have explained the method which should be adopted in order to get rid of this when we know the actual increase. The accuracy of the Chester observations is such as to make it desirable to take into account this circumstance. The births in all England in the year 1700, according to the Parliamentary Reports, were 138,979, and in 1780, 201,310, making the mean annual rate of increase 1.0046; in the county of Chester, taken by itself, in 1700, they were 2650, and in 1780, 4592, making the mean

annual rate of increase 1.0061. We may, therefore, suppose the births to have increased in geometrical progression, during the century previous to Dr. Haygarth's observations, at the rate of 1.005, without fear of an error which can disturb the accuracy of our results. The deaths at the same time were about equal to the deaths forty years previously, a result which is confirmed by direct calculation. These data are sufficient to correct the table of mortality, and it is obvious that the error of our hypothesis is altogether of an order to be neglected, for a small inequality in the rate of increase will not affect the result, unless it be of long period.

91. Table (1), page 56, contains the observations of deaths by Dr. Haygarth at Chester.

Table (2) has been calculated upon this hypothesis, namely, that the births increased during the century previous to Dr. Haygarth's observation in a geometric progression, of which the common ratio was 1.005.

Table (3) shows the values of annuities on such lives.

Table (4) shows the values of single premiums for insuring £1. payable at death; and

Table (5) shows the values of annual premiums for insuring £1. payable at death.

Tables (3), (4), and (5), have been calculated from Table (2) by Mr. David Jones of the Royal Exchange Assurance Company.

92. Mr. Finlaison has recently published very extensive tables of mortality, formed from the government tontines and annuitants, which are rendered extremely valuable by the accuracy of the materials from which they have been deduced, and by the very great care and attention which have been bestowed on them by the author.

Mr. Finlaison (in his valuable report to the Lords of the Treasury) explains at length the manner in which he made use of the records of the tontines. Mr. Finlaison observes, "that the facts shown in these observations, bear conclusive testimony that the rate of mortality in England has, during the last century, diminished in a very important degree, on each sex equally, but not by equal gradations, nor equally at all periods of life; and that while in regard to the males it seems in early and middling life to have remained for a long time as it stood about fifty years ago, in respect of the females it has during the same time visibly and progressively diminished to this day by slight but still sensible gradations."

Whether life has generally improved or not, it is certain that epidemics are of much less frequent occurrence now than they were formerly, which circumstance must have a very favourable influence.

93. Mr. Griffith Davies has published tables of annuities taken from statements of Mr. Morgan, in his addresses to the general courts of the Equitable Society, and in notes added by him to the latter editions of Dr. Price's "Observations on Reversionary Payments." In Mr. Morgan's address to the general court held on the 24th April, 1800, he stated that the decrements of life among the members of the Equitable for the preceding thirty years, had been to those of the Northampton

from 10 to 20 as 1 to 2	from 40 to 50 as 3 to 5
20 .. 30 .. 1 .. 2	50 .. 60 .. 5 .. 7
30 .. 40 .. 3 .. 5	60 .. 80 .. 4 .. 5

which statement is confirmed in his subsequent addresses. In a recent publication, Mr. Morgan admits that he was not then aware of the great number of instances in which there are several policies upon one and the same life, but this circumstance cannot very materially affect Mr. Davies's

calculations.' Such statements as these, although not so detailed as might be wished, sufficiently prove that in the Equitable Society, the rate of mortality is considerably less than that given by the Northampton Table.

94. Mr. Babbage, in a work entitled "A Comparative View of the various Institutions for the Assurance of Lives," has examined the advantages which are presented by the different insurance offices in this metropolis. It is not our intention to follow him in this inquiry, which is rendered very intricate from the complicated manner in which some of the offices make returns to the assured of a portion of the immense profits which they accumulate, instead of charging, which is obviously a simpler method, the real value at first.

The offices which use the Northampton Table as the basis of their calculations are the

Albion,	Law Life,
Atlas,	London Life Association,
Eagle,	Pelican,
Exchange, Royal,	Provident,
Globe,	Rock,
Imperial,	Westminster.

95. The doctrine of fire and sea insurances seems to be at present nearly in the same state in which that of life insurances was at the beginning of the last century. Montucla mentions a treatise on the subject of ship insurances by Montandouin, a merchant of Nantes, of which he speaks in terms of commendation, and seems to intimate that the publication of this work drew the attention of the Académie des Sciences. That learned body proposed the theory of maritime insurances as a prize question in 1783, 1785, and in 1787, but without much success. None of the essays received were thought to deserve the prize, but, on the last occasion, half the prize of 6000 livres was divided between Lacroix and Biquilly, two of the competitors. The remaining 3000 livres were intended to be offered, in 1791, for the best tables of premiums for maritime insurance, but the revolution intervened to prevent any adjudication of it. All our present knowledge on this subject seems to be confined to the personal experience of the underwriters.

96. Another extensive application of the theory of probability has been made by Condorcet, at the instance of the enlightened financier Turgot. In a work entitled "Essai sur la Probabilité des Décisions," Condorcet has investigated and compared the probabilities of error in the decisions pronounced by tribunnals more or less numerous, and various schemes for determining the verdict. Connected with the same question is the inquiry into the best mode of collecting votes in elections, in which more than two conflicting propositions are presented to each elector. Condorcet has examined in detail the respective advantages and disadvantages of electing by a simple majority, by a majority exceeding a given number, or by a number proportional to the whole number of voters, with many others. He arrives at the conclusion that the best mode of electing is by a majority not below a given number of a single assembly.

TABLE I.—DR. HAYGARTH'S OBSERVATIONS at CHESTER, as given by Dr. Price in his work on Reversionary Payments, Vol. ii.

Age.	MALES.		FEMALES.		Age.	MALES.		FEMALES.	
		Deaths.		Deaths.			Deaths.		Deaths.
0	1927	438	2139	368	50	558	16	752	15
1	1489	180	1771	181	51	542	16	737	14
2	1309	107	1580	127	52	526	16	723	14
3	1202	67	1463	77	53	519	16	709	14
4	1135	34	1386	53	54	494	15	695	14
5	1101	30	1333	30	55	479	14	681	13
6	1071	24	1303	10	56	465	14	668	13
7	1047	18	1285	11	57	451	14	655	13
8	1029	11	1274	9	58	437	14	642	15
9	1018	8	1265	7	59	423	16	627	15
10	1010	6	1258	6	60	407	19	612	20
11	1004	5	1252	6	61	388	22	592	25
12	999	5	1246	7	62	366	22	567	25
13	994	6	1239	7	63	344	22	542	25
14	988	6	1232	8	64	322	20	517	21
15	982	7	1224	9	65	302	16	496	17
16	975	9	1215	10	66	286	13	479	15
17	966	10	1205	11	67	273	11	464	15
18	956	11	1194	12	68	262	11	449	16
19	945	11	1182	11	69	251	13	433	20
20	934	11	1171	10	70	238	16	413	25
21	923	11	1161	10	71	222	22	388	30
22	912	12	1151	10	72	200	22	358	30
23	900	12	1141	11	73	178	21	328	30
24	888	12	1130	12	74	157	18	298	27
25	876	13	1118	16	75	139	15	271	23
26	863	13	1102	16	76	124	12	248	22
27	850	13	1086	16	77	112	11	226	21
28	837	12	1070	16	78	101	11	205	21
29	825	11	1054	16	79	90	10	184	21
30	814	10	1038	13	80	80	10	163	21
31	804	9	1025	13	81	70	10	142	21
32	795	10	1012	13	82	60	9	121	21
33	785	10	999	13	83	51	8	100	21
34	775	10	986	13	84	43	7	79	18
35	765	11	973	14	85	36	6	61	12
36	754	11	959	14	86	30	5	49	8
37	743	12	945	14	87	25	4	41	6
38	731	12	931	14	88	21	4	35	4
39	719	13	917	15	89	17	3	31	4
40	706	13	902	15	90	14	3	27	4
41	693	14	887	15	91	11	3	23	4
42	679	14	872	15	92	8	3	19	4
43	665	15	857	14	93	5	2	15	4
44	650	15	843	15	94	3	2	11	4
45	635	15	828	15	95	1	1	7	3
46	620	15	813	15	96			4	3
47	605	15	798	15	97			1	1
48	590	16	783	16					
49	574	16	767	15					

TABLE II.—TABLE of MORTALITY formed from the Observations of Dr. Haygarth at CHES-
TER, corrected for the increase of Population during the Century previous to the Observa-
tions upon the supposition that the Births increased yearly in a geometrical progression of
which the common ratio was 1.005.

Age.	MALES.		FEMALES.		Age.	MALES.		FEMALES.	
	Living.	Deaths.	Living.	Deaths.		Living.	Deaths.	Living.	Deaths.
0	10000	1778	10000	1351	50	3675	92	4302	77
1	8222	739	8649	670	51	3583	93	4225	72
2	7483	445	7979	474	52	3490	94	4153	73
3	7038	283	7505	291	53	3396	94	4080	73
4	6755	149	7214	203	54	3302	89	4007	73
5	6606	133	7011	118	55	3213	84	3934	69
6	6473	108	6593	73	56	3129	84	3865	69
7	6365	84	6820	47	57	3045	85	3796	70
8	6281	55	6773	40	58	2960	85	3726	80
9	6266	42	6733	32	59	2875	97	3646	80
10	6184	35	6701	28	60	2778	113	3566	105
11	6149	30	6673	29	61	2665	131	3461	130
12	6119	30	6644	33	62	2534	131	3331	131
13	6089	35	6611	33	63	2403	131	3200	132
14	6054	34	6578	37	64	2272	121	3068	112
15	6020	40	6541	41	65	2151	100	2956	93
16	5980	48	6500	45	66	2051	83	2863	83
17	5932	53	6455	49	67	1968	72	2750	83
18	5879	57	6406	54	68	1896	72	2697	89
19	5822	57	6352	50	69	1824	84	2608	110
20	5765	58	6302	46	70	1740	102	2498	136
21	5707	59	6256	46	71	1638	137	2362	163
22	5648	63	6210	46	72	1501	137	2199	164
23	5585	63	6164	51	73	1364	132	2035	165
24	5522	63	6113	55	74	1232	116	1870	150
25	5459	69	6058	72	75	1116	98	1720	129
26	5390	69	5986	72	76	1018	81	1591	124
27	5321	69	5914	73	77	937	76	1467	120
28	5252	64	5841	73	78	861	76	1347	120
29	5188	61	5768	73	79	785	70	1227	121
30	5127	56	5695	61	80	715	71	1106	121
31	5071	51	5634	61	81	644	71	985	122
32	5020	57	5573	62	82	573	65	863	123
33	4963	57	5511	62	83	508	59	740	123
34	4906	57	5449	62	84	449	54	617	107
35	4849	62	5387	67	85	395	47	510	74
36	4787	62	5320	67	86	348	42	436	52
37	4725	68	5253	67	87	306	35	384	41
38	4657	68	5186	68	88	271	36	343	30
39	4589	73	5118	73	89	235	30	313	30
40	4516	73	5045	73	90	205	29	283	31
41	4443	79	4972	73	91	176	30	252	31
42	4364	79	4899	73	92	146	30	221	31
43	4285	84	4826	69	93	116	24	190	32
44	4201	85	4757	74	94	92	24	158	32
45	4116	85	4683	75	95	68	17	126	26
46	4031	86	4608	75	96	51	7	100	26
47	3945	86	4533	75	97	44	7	74	14
48	3859	92	4458	80	98	37	7	60	8
49	3767	92	4378	76	99	30	7	52	8

TABLE III.—PRESENT VALUE of an ANNUITY of 1*l.* payable during the Life of a Male or Female. CHESTER RATE of MORTALITY.

Age.	3 per cent. $a_m, r = \frac{1}{1.03}.$		4 per cent. $a_m, r = \frac{1}{1.04}.$		5 per cent. $a_m, r = \frac{1}{1.05}.$		6 per cent. $a_m, r = \frac{1}{1.06}.$		Age.
	Males.	Females.	Males.	Females.	Males.	Females.	Males.	Females.	
0	16.0427	17.7180	13.3465	14.6363	11.3697	12.4024	9.8755	10.7291	0
1	19.0973	20.1002	15.8820	16.5994	13.5198	14.0566	11.7318	12.1496	1
2	20.6128	21.4417	17.1484	17.7130	14.5978	14.9988	12.6638	12.9597	2
3	21.5735	22.4798	17.9620	18.5850	15.2968	15.7434	13.2723	13.6049	3
4	22.1517	23.0882	18.4631	19.1081	15.7346	16.1974	13.6581	14.0030	4
5	22.3308	23.4694	18.6347	19.4478	15.8939	16.4997	13.8041	14.2729	5
6	22.4734	23.5873	18.7783	19.5720	16.0315	16.6213	13.9330	14.3883	6
7	22.5404	23.5549	18.8608	19.5727	16.1187	16.6392	14.0196	14.4149	7
8	22.5271	23.4299	18.8776	19.4969	16.1510	16.5924	14.0595	14.3858	8
9	22.4078	23.2762	18.8061	19.3972	16.1084	16.5255	14.0348	14.3395	9
10	22.2368	23.0890	18.6912	19.2694	16.0286	16.4346	13.9779	14.2725	10
11	22.0343	22.8815	18.5495	19.1243	15.9259	16.3288	13.9009	14.1923	11
12	21.8066	22.6708	18.3961	18.9761	15.8042	16.2200	13.8072	14.1095	12
13	21.5715	22.4675	18.2157	18.8336	15.6761	16.1160	13.7077	14.0307	13
14	21.3471	22.2576	18.0539	18.6852	15.5551	16.0067	13.6142	13.9472	14
15	21.1117	22.0549	17.8821	18.5426	15.4251	15.9021	13.5125	13.8676	15
16	20.8905	21.8599	17.7217	18.4059	15.3047	15.8026	13.4191	13.7924	16
17	20.6913	21.6727	17.5797	18.2756	15.1999	15.7084	13.3393	13.7218	17
18	20.5041	21.4936	17.4478	18.1520	15.1038	15.6200	13.2672	13.6564	18
19	20.3260	21.3266	17.3233	18.0385	15.0143	15.5404	13.2009	13.5989	19
20	20.1428	21.1407	17.1944	17.9089	14.9209	15.4468	13.1313	13.5292	20
21	19.9580	20.9350	17.0639	17.7622	14.8261	15.3385	13.0606	13.4464	21
22	19.7714	20.7228	16.9318	17.6096	14.7301	15.2247	12.9889	13.3588	22
23	19.5943	20.5038	16.8077	17.4506	14.6410	15.1052	12.9235	13.2660	23
24	19.4124	20.2951	16.6795	17.3001	14.5485	14.9928	12.8552	13.1792	24
25	19.2255	20.0937	16.5469	17.1554	14.4522	14.8854	12.7838	13.0968	25
26	19.0558	19.9455	16.4290	17.0562	14.3691	14.8176	12.7243	13.0496	26
27	18.8819	19.7939	16.3078	16.9544	14.2831	14.7479	12.6626	13.0010	27
28	18.7039	19.6426	16.1829	16.8530	14.1943	14.6789	12.5987	12.9533	28
29	18.5027	19.4879	16.0378	16.7489	14.0879	14.6079	12.5194	12.9042	29
30	18.2845	19.3298	15.8778	16.6422	13.9683	14.5349	12.4285	12.8538	30
31	18.0410	19.1253	15.6952	16.4953	13.8287	14.4268	12.3196	12.7726	31
32	17.7710	18.9147	15.4889	16.3428	13.6677	14.3140	12.1915	12.6871	32
33	17.5144	18.7013	15.2934	16.1878	13.5158	14.1988	12.0714	12.5996	33
34	17.2495	18.4815	15.0900	16.0268	13.3565	14.0784	11.9444	12.5076	34
35	16.9758	18.2550	14.8780	15.8597	13.1892	13.9524	11.8098	12.4106	35
36	16.7115	18.0395	14.6736	15.7019	13.0260	13.8345	11.6806	12.3210	36
37	16.4387	17.8177	14.4608	15.5382	12.8589	13.7115	11.5439	12.2268	37
38	16.1791	17.5893	14.2588	15.3685	12.6990	13.5831	11.4152	12.1278	38
39	15.9114	17.3577	14.0489	15.1956	12.5315	13.4518	11.2794	12.0263	39
40	15.6537	17.1371	13.8470	15.0321	12.3708	13.3287	11.1494	11.9321	40
41	15.3882	16.9104	13.6375	14.8629	12.2028	13.2007	11.0126	11.8340	41
42	15.1368	16.6772	13.4398	14.6878	12.0449	13.0672	10.8846	11.7310	42
43	14.8783	16.4374	13.2350	14.5064	11.8803	12.9281	10.7504	11.6291	43
44	14.6311	16.1761	13.0397	14.3054	11.7237	12.7714	10.6233	11.4990	44
45	14.3819	15.9246	12.8413	14.1128	11.5641	12.6219	10.4933	11.3815	45
46	14.1250	15.6693	12.6366	13.9162	11.3984	12.4687	10.3574	11.2608	46
47	13.8659	15.4064	12.4285	13.7123	11.2292	12.3088	10.2182	11.1339	47
48	13.6002	15.1356	12.2137	13.5007	11.0534	12.1416	10.0726	11.0005	48
49	13.3503	14.8746	12.0125	13.2973	10.8895	11.9817	9.9378	10.8736	49

TABLE III.—continued.

Age. m	3 per cent. $a_m, r = \frac{1}{1.03}$		4 per cent. $a_m, r = \frac{1}{1.04}$		5 per cent. $a_m, r = \frac{1}{1.05}$		6 per cent. $a_m, r = \frac{1}{1.06}$		Age. m
	Males.	Females.	Males.	Females.	Males.	Females.	Males.	Females.	
50	13-0950	14-5914	11-8057	13-0735	10-7202	11-8030	9-7978	10-7297	50
51	12-8342	14-3031	11-5932	12-8442	10-5453	11-6190	9-6523	10-5807	51
52	12-5715	13-9876	11-3782	12-5895	10-3676	11-4115	9-5041	10-4100	52
53	12-3070	13-6650	11-1609	12-3274	10-1873	11-1964	9-3531	10-2320	53
54	12-0371	13-3314	10-9378	12-0541	10-0012	10-9704	9-1966	10-0435	54
55	11-7417	12-9861	10-6904	11-7688	9-7921	10-7327	9-0184	9-8437	55
56	11-4186	12-6145	10-4166	11-4581	9-5577	10-4705	8-8161	9-6206	56
57	11-0856	12-2291	10-1320	11-1330	9-3125	10-1939	8-6029	9-3832	57
58	10-7460	11-8326	9-8399	10-7959	9-0589	9-9047	8-3809	9-1339	58
59	10-3956	11-4550	9-5360	10-4741	8-7930	9-6281	8-1464	8-6935	59
60	10-0814	11-0633	9-2637	10-1374	8-5551	9-3363	7-9368	8-6386	60
61	9-8241	10-7409	9-0428	9-8628	8-3637	9-1005	7-7697	8-4347	61
62	9-6420	10-4949	8-8907	9-6576	8-2359	8-9285	7-6616	8-2897	62
63	9-4726	10-2523	8-7504	9-4551	8-1191	8-7587	7-5641	8-1468	63
64	9-3194	10-0142	8-6251	9-2563	8-0166	8-5923	7-4802	8-0072	64
65	9-1389	9-7055	8-4747	8-9913	7-8910	8-3638	7-3751	7-8092	65
66	8-8720	9-3213	8-2434	8-6547	7-6895	8-0672	7-1987	7-5466	66
67	8-5236	8-8876	7-9347	8-2696	7-4145	7-7235	6-9525	7-2382	67
68	8-1127	8-4360	7-5655	7-8651	7-0809	7-3592	6-6495	6-9086	68
69	7-6859	7-9856	7-1787	7-4588	6-7284	6-9909	6-3266	6-5731	69
70	7-2987	7-5874	6-8263	7-0988	6-4059	6-6637	6-0300	6-2743	70
71	6-9858	7-2649	6-5414	6-8078	6-1450	6-3997	5-7898	6-0337	71
72	6-6521	7-0376	6-2420	6-6050	6-0412	6-2178	5-6974	5-8698	72
73	6-7665	6-8328	6-3520	6-4227	5-9803	6-0548	5-6458	5-7234	73
74	6-7163	6-6588	6-3139	6-2690	5-9521	5-9185	5-6257	5-6021	74
75	6-6368	6-4567	6-2489	6-0884	5-8994	5-7564	5-5831	5-4561	75
76	6-4940	6-1897	6-1245	5-8153	5-7907	5-5313	5-4878	5-2524	76
77	6-2670	5-9142	5-9201	5-5930	5-6058	5-3022	5-3200	5-0381	77
78	6-0248	5-6343	5-7004	5-3349	5-4056	5-0633	5-1369	4-8162	78
79	5-8063	5-3709	5-5024	5-0909	5-2254	4-8364	4-9723	4-6044	79
80	5-5660	5-1373	5-2827	4-8738	5-0239	4-6312	4-7867	4-4146	80
81	5-3651	4-9414	5-0997	4-6914	4-8568	4-4632	4-6333	4-2544	81
82	5-2108	4-8092	4-9609	4-5687	4-7313	4-3488	4-5198	4-1471	82
83	5-0538	4-7768	4-8195	4-5413	4-6035	4-3253	4-4040	4-1266	83
84	4-8894	4-9009	4-6709	4-6644	4-4689	4-4469	4-2817	4-2462	84
85	4-7246	5-1070	4-5218	4-8688	4-3338	4-6488	4-1591	4-4453	85
86	4-5236	5-1530	4-3379	4-9229	4-1651	4-7095	4-0040	4-5118	86
87	4-2988	5-0263	4-1306	4-8132	3-9736	4-6149	3-8268	4-4302	87
88	3-9996	4-7960	3-8506	4-6041	3-7111	4-4249	3-5803	4-2573	88
89	3-7507	4-4133	3-6181	4-2471	3-4936	4-0914	3-3765	3-9453	89
90	3-4285	4-0275	3-3135	3-8853	3-2051	3-7514	3-1029	3-6253	90
91	3-1133	3-6587	3-0138	3-5378	2-9199	3-4235	2-8310	3-3156	91
92	2-8656	3-2971	2-7785	3-1954	2-6959	3-0989	2-6175	3-0075	92
93	2-7149	2-9561	2-6369	2-8654	2-5627	2-7848	2-2921	2-7080	93
94	2-5258	2-6539	2-4578	2-5835	2-3928	2-5162	2-3307	2-4519	94
95	2-5195	2-4278	2-4582	2-3692	2-3992	2-3130	2-3425	2-2591	95
96	2-4605	2-1508	2-4088	2-1047	2-3589	2-0601	2-3107	2-0172	96
97	1-9375	1-9937	1-9037	1-9579	1-8708	1-9232	1-8300	1-8896	97
98	1-3731	1-5327	1-3544	1-5113	1-3360	1-4905	1-3182	1-4703	98
99	7-443	8-215	7-372	8-136	7-302	8-059	7-233	7-983	99

TABLE IV.—SINGLE PREMIUM required to secure the payment of 1*l*. at the end of the year in which the life shall fail.

Age.	3 per cent.		4 per cent.		5 per cent.		Age.
	Males.	Females.	Males.	Females.	Males.	Females.	
0	•50361	•45482	•44821	•39860	•41097	•36179	0
1	•41464	•38543	•35070	•32310	•30858	•28302	1
2	•37050	•34636	•30198	•28027	•25725	•23815	2
3	•34252	•31612	•27069	•24673	•22396	•20270	3
4	•32568	•29840	•25142	•22661	•20311	•18108	4
5	•32046	•28730	•24482	•21355	•19553	•16668	5
6	•31631	•28387	•23930	•20877	•18898	•16089	6
7	•31436	•28481	•23612	•20874	•18482	•16004	7
8	•31475	•28845	•23548	•21166	•18329	•16227	8
9	•31822	•29293	•23823	•21549	•18531	•16545	9
10	•32320	•29838	•24265	•22041	•18911	•16978	10
11	•32910	•30442	•24810	•22599	•19400	•17482	11
12	•33573	•31056	•25438	•23169	•19960	•18000	12
13	•34258	•31648	•26093	•23717	•20590	•18495	13
14	•34911	•32259	•26716	•24288	•21166	•19016	14
15	•35597	•32850	•27377	•24836	•21785	•19514	15
16	•36241	•33418	•27993	•25362	•22359	•19988	16
17	•36822	•33963	•28540	•25863	•22858	•20436	17
18	•37367	•34485	•29047	•26338	•23315	•20857	18
19	•37885	•34971	•29526	•26775	•23741	•21236	19
20	•38419	•35513	•30022	•27273	•24186	•21682	20
21	•38957	•36112	•30523	•27838	•24638	•22198	21
22	•39501	•36730	•31032	•28425	•25095	•22740	22
23	•40017	•37368	•31509	•29036	•25519	•23308	23
24	•40546	•37975	•32002	•29615	•25960	•23844	24
25	•41091	•38562	•32512	•30172	•26418	•24355	25
26	•41585	•38994	•32965	•30553	•26814	•24678	26
27	•42092	•39435	•33432	•30945	•27223	•25010	27
28	•42610	•39876	•33912	•31335	•27646	•25339	28
29	•43196	•40327	•34470	•31735	•28153	•25677	29
30	•43832	•40787	•35085	•32145	•28722	•26024	30
31	•44541	•41383	•35788	•32710	•29387	•26539	31
32	•45327	•41996	•36581	•33297	•30154	•27076	32
33	•46075	•42618	•37333	•33893	•30877	•27625	33
34	•46846	•43258	•38115	•34512	•31636	•28198	34
35	•47643	•43917	•38931	•35155	•32432	•28798	35
36	•48413	•44545	•39717	•35762	•33200	•29360	36
37	•49208	•45191	•40535	•36392	•34005	•29945	37
38	•49964	•45856	•41312	•37044	•34767	•30557	38
39	•50744	•46531	•42120	•37709	•35564	•31182	39
40	•51494	•47174	•42896	•38338	•36330	•31768	40
41	•52267	•47834	•43702	•38989	•37130	•32378	41
42	•53000	•48513	•44462	•39662	•37881	•33013	42
43	•53753	•49211	•45250	•40360	•38665	•33676	43
44	•54473	•49973	•46001	•41133	•39411	•34422	44
45	•55200	•50705	•46764	•41874	•40171	•35134	45
46	•55947	•51449	•47552	•42630	•40960	•35863	46
47	•56701	•52214	•48352	•43414	•41766	•36625	47
48	•57475	•53003	•49178	•44228	•42603	•37421	48
49	•58203	•53763	•49952	•45010	•43383	•38182	49

TABLE IV.—continued.

Age.	3 per cent.		4 per cent.		5 per cent.		Age.
m	Males.	Females.	Males.	Females.	Males.	Females.	m
50	58947	54588	50747	45871	44190	39033	50
51	59706	55428	51565	46753	45022	39910	51
52	60471	56347	52392	47733	45869	40898	52
53	61242	57286	53227	48741	46727	41922	53
54	62028	58258	54085	49792	47613	42998	54
55	62888	59264	55037	50889	48610	44130	55
56	63829	60346	56090	52084	49725	45379	56
57	64799	61469	57185	53335	50893	46696	57
58	65788	62623	58308	54631	52100	48073	58
59	66809	63723	59477	55869	53367	49390	59
60	67724	64864	60524	57164	54500	50780	60
61	68473	65803	61374	58220	55411	51902	61
62	69004	66520	61959	59009	56020	52721	62
63	69497	67226	62498	59788	56576	53530	63
64	69943	67920	62980	60553	57064	54322	64
65	70469	68619	63559	61572	57662	55410	65
66	71247	69938	64448	62867	58621	56823	66
67	72261	71201	65636	64348	59931	58460	67
68	73458	72517	67056	65903	61520	60194	68
69	74701	73828	68543	67466	63198	61948	69
70	75829	74988	69899	68851	64734	63506	70
71	76740	75927	70995	69970	65976	64763	71
72	77130	76590	71446	70750	66470	65630	72
73	77379	77186	71723	71451	66760	66406	73
74	77525	77693	71870	72042	66895	67055	74
75	77757	78281	72120	72737	67146	67827	75
76	78173	79059	72598	73672	67663	68884	76
77	78834	79862	73384	74642	68544	69990	77
78	79539	80677	74229	75635	69497	71127	78
79	80176	81444	74991	76573	70355	72208	79
80	80876	82124	75836	77408	71315	73170	80
81	81461	82695	76540	78110	72110	73985	81
82	81910	83080	77073	78582	72708	74530	82
83	82368	83174	77617	78687	73317	74641	83
84	82846	82813	78189	78214	73958	74062	84
85	83326	82213	78762	77428	74601	73101	85
86	83912	82079	79470	77220	75404	72810	86
87	84567	82448	80267	77642	76316	73262	87
88	85438	83118	81344	78446	77566	74167	88
89	86163	84233	82238	79819	78602	75735	89
90	87101	85357	83409	81210	79976	77374	90
91	88020	86431	84562	82547	81333	78936	91
92	88741	87484	85467	83864	82400	80481	92
93	89180	88477	86012	85133	83035	81978	93
94	89731	89358	86701	86217	83841	83256	94
95	89748	90016	86699	87042	83813	84224	95
96	89921	90823	86889	88059	84005	85428	96
97	91444	91280	88832	88623	86330	86080	97
98	93088	92523	90945	90341	88876	88140	98
99	94920	94695	93318	93025	91761	91400	99

TABLE V.—ANNUAL PREMIUM required to secure *1l.* at the end of the year in which the life shall fail.

Age.	3 per cent.		4 per cent.		5 per cent.		Age.
	Males.	Females.	Males.	Females.	Males.	Females.	
0	·02955	·02429	·03124	·02549	·03322	·02699	0
1	·02063	·01826	·02078	·01836	·02125	·01880	1
2	·01714	·01543	·01664	·01498	·01649	·01459	2
3	·01517	·01346	·01428	·01260	·01374	·01211	3
4	·01406	·01238	·01292	·01127	·01214	·01053	4
5	·01373	·01174	·01247	·01045	·01157	·00952	5
6	·01347	·01154	·01210	·01015	·01109	·00913	6
7	·01335	·01160	·01189	·01015	·01080	·00907	7
8	·01337	·01180	·01185	·01033	·01069	·00922	8
9	·01359	·01206	·01203	·01057	·01083	·00944	9
10	·01390	·01238	·01232	·01088	·01110	·00974	10
11	·01428	·01274	·01269	·01123	·01146	·01009	11
12	·01472	·01312	·01312	·01160	·01189	·01045	12
13	·01517	·01348	·01358	·01196	·01235	·01080	13
14	·01562	·01387	·01402	·01234	·01278	·01118	14
15	·01609	·01424	·01450	·01271	·01326	·01154	15
16	·01655	·01461	·01495	·01307	·01371	·01189	16
17	·01697	·01498	·01536	·01342	·01411	·01223	17
18	·01737	·01533	·01575	·01375	·01448	·01255	18
19	·01776	·01566	·01612	·01406	·01482	·01284	19
20	·01817	·01604	·01650	·01443	·01519	·01318	20
21	·01858	·01646	·01690	·01484	·01557	·01359	21
22	·01901	·01690	·01731	·01528	·01595	·01401	22
23	·01943	·01737	·01770	·01574	·01631	·01447	23
24	·01986	·01783	·01810	·01618	·01670	·01491	24
25	·02031	·01828	·01853	·01662	·01710	·01533	25
26	·02073	·01861	·01892	·01692	·01745	·01560	26
27	·02117	·01896	·01932	·01723	·01781	·01588	27
28	·02162	·01931	·01974	·01755	·01819	·01616	28
29	·02215	·01968	·02023	·01788	·01866	·01645	29
30	·02273	·02006	·02079	·01822	·01919	·01675	30
31	·02339	·02056	·02144	·01870	·01982	·01720	31
32	·02414	·02108	·02219	·01920	·02056	·01768	32
33	·02488	·02163	·02292	·01972	·02127	·01817	33
34	·02567	·02220	·02369	·02027	·02204	·01870	34
35	·02650	·02280	·02452	·02085	·02286	·01926	35
36	·02733	·02339	·02534	·02141	·02367	·01979	36
37	·02821	·02401	·02622	·02202	·02454	·02036	37
38	·02908	·02466	·02708	·02263	·02538	·02095	38
39	·03000	·02534	·02799	·02329	·02628	·02158	39
40	·03092	·02601	·02889	·02392	·02717	·02217	40
41	·03189	·02670	·02986	·02458	·02812	·02280	41
42	·03284	·02744	·03079	·02528	·02904	·02347	42
43	·03385	·02822	·03179	·02603	·03002	·02418	43
44	·03484	·02909	·03277	·02688	·03097	·02499	44
45	·03588	·02996	·03379	·02771	·03197	·02579	45
46	·03699	·03086	·03487	·02858	·03304	·02663	46
47	·03814	·03182	·03601	·02951	·03415	·02752	47
48	·03936	·03284	·03722	·03050	·03534	·02847	48
49	·04056	·03386	·03839	·03148	·03649	·02941	49

TABLE V.—continued.

Age. m	3 per cent.		4 per cent.		5 per cent.		Age. m
	Males.	Females.	Males.	Females.	Males.	Females.	
50	·04182	·03501	·03963	·03260	·03770	·03049	50
51	·04316	·03622	·04095	·03377	·03900	·03463	51
52	·04455	·03759	·04233	·03513	·04035	·03295	52
53	·04602	·03906	·04377	·03657	·04177	·03437	53
54	·04757	·04065	·04531	·03814	·04328	·03592	54
55	·04935	·04237	·04708	·03986	·04504	·03761	55
56	·05139	·04432	·04913	·04181	·04710	·03956	56
57	·05361	·04646	·05137	·04396	·04935	·04171	57
58	·05600	·04880	·05379	·04632	·05180	·04408	58
59	·05862	·05116	·05645	·04869	·05449	·04647	59
60	·06111	·05377	·05897	·05133	·05704	·04913	60
61	·06326	·05604	·06111	·05360	·05917	·05139	61
62	·06484	·05786	·06265	·05537	·06066	·05310	62
63	·06636	·05974	·06410	·05719	·06204	·05485	63
64	·06778	·06166	·06543	·05904	·06329	·05663	64
65	·06950	·06428	·06708	·06163	·06485	·05918	65
66	·07217	·06776	·06973	·06512	·06746	·06267	66
67	·07587	·07201	·07346	·06942	·07122	·06701	67
68	·08060	·07685	·07829	·07434	·07613	·07201	68
69	·08600	·08216	·08381	·07976	·08177	·07752	69
70	·09137	·08732	·08931	·08502	·08741	·08287	70
71	·09609	·09187	·09414	·08962	·09234	·09752	71
72	·09822	·09529	·09624	·09303	·09440	·09093	72
73	·09963	·09854	·09756	·09626	·09564	·09413	73
74	·10047	·10144	·09827	·09911	·09622	·09692	74
75	·10182	·10498	·09949	·10262	·09732	·10039	75
76	·10431	·10996	·10190	·10763	·09964	·10542	76
77	·10848	·11550	·10605	·11322	·10376	·11105	77
78	·11322	·12160	·11078	·11940	·10849	·11731	78
79	·11779	·12783	·11533	·12572	·11301	·12372	79
80	·12317	·13381	·12071	·13179	·11839	·12988	80
81	·12798	·13918	·12548	·13724	·12313	·13542	81
82	·13188	·14301	·12930	·14111	·12686	·13934	82
83	·13606	·14398	·13338	·14200	·13084	·14016	83
84	·14067	·14034	·13788	·13808	·13523	·13597	84
85	·14555	·13462	·14264	·13193	·13986	·12941	85
86	·15191	·13339	·14888	·13037	·14599	·12752	86
87	·15960	·13681	·15645	·13356	·15344	·13048	87
88	·17089	·14340	·16770	·13998	·16464	·13672	88
89	·18137	·15560	·17808	·15212	·17492	·14879	89
90	·19668	·16977	·19337	·16624	·19019	·16284	90
91	·21399	·18532	·21068	·18191	·20749	·17845	91
92	·22957	·20359	·22620	·19990	·22295	·19635	92
93	·24006	·22403	·23650	·22025	·23306	·21660	93
94	·25450	·24455	·25074	·24059	·24712	·23678	94
95	·25498	·26260	·25071	·25834	·24657	·25422	95
96	·25985	·28825	·25490	·28363	·25010	·27916	96
97	·31130	·30490	·30593	·29962	·30071	·29447	97
98	·39227	·36571	·38628	·35973	·38046	·35390	98
99	·54417	·51987	·53719	·77515	·53036	·50613	99

The following tables will serve to show how nearly different tables of mortality and different tables of annuities agree.

Tables of Mortality.

Age.	(1.) Despar- cieux.	(2.) North- ampton, Dr. Price.	(3.) Carlisle, Mr. Milne.	(4.) Equita- ble Ex- perience, Mr. Davies.	(5.) Mr. Finlaison's Tables.		(6.) Chester Table.		Age.
	Living.	Living.	Living.	Living.	Male.	Female.	Male.	Female.	
0		1000	1000		1000	1000	10000	10000	0
10	830	487	646	2844	896	903	6184	6701	10
20	814	440	609	2705	837	848	5765	6302	20
30	734	415	564	2501	732	777	5127	5695	30
40	657	312	507	2236	644	700	4516	5045	40
50	581	245	439	1937	561	623	3675	4302	50
60	463	174	364	1524	440	539	2778	3566	60
70	310	105	240	1028	288	412	1740	2498	70
80	118	40	95	480	125	210	715	617	80
90	11	3	14	65	11	52	205	283	90

Annuities at 3 per cent.

Age.	(1.) Despar- cieux.	(2.) North- ampton, Dr. Price.	(3.) Carlisle, Mr. Milne.	(4.) Equita- ble Ex- perience, Mr. Davies.	(5.) Mr. Finlaison's Tables.		(6.) Chester Table.		Age.
					Male.	Female.	Male.	Female.	
0			17.320				16.042	17.718	0
10	22.766	20.663	23.512	23.768	20.524	22.312	22.237	23.089	10
20	21.168	18.638	21.694	21.795	19.223	20.720	20.143	21.140	20
30	19.492	16.922	19.556	19.671	17.082	19.174	18.284	19.330	30
40	17.183	14.848	17.143	17.351	14.011	16.111	15.654	17.137	40
50	13.899	12.436	14.303	14.477	10.777	12.560	13.095	14.591	50
60	10.522	9.777	10.491	11.539	7.425	8.638	10.081	11.063	60
70		6.734	7.123	8.285			7.299	7.587	70

These numbers are extracted from the following works:—

- (1.) The Doctrine of Life Annuities, by Mr. Francis Bailey, vol. ii. p. 532.
- (2.) Observations on Reversionary Payments, by Dr. Price, vol. ii. p. 314.
- (3.) A Treatise on the Valuation of Annuities, by Mr. Milne, vol. ii. p. 394.
- (4.) Tables of Life Contingencies, by Mr. Griffith Davies.
- (5.) These tables are taken from the Report of the House of Commons on Friendly Societies, 1827.
- (6.) These values are extracted from Table III. p. 58.

The Table of Mr. Milne agrees very closely with the mean of the columns for males and females in Table (1), and, in fact, it would have been sufficient to have supposed the rate of increase 1.007 in the formation of Table (2) during the last century to render the difference almost insensible.